Equivalence of Borcherds G-Vertex Algebras and Axiomatic Vertex Algebras

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Abstract

In this paper we build an abstract description of vertex algebras from their basic axioms. Starting with Borcherds' notion of a vertex group, we naturally construct a family of multilinear singular maps parameterised by trees. These singular maps are defined in a way which focusses on the relations of singularities to their inputs. In particular we show that this description of a vertex algebra allows us to present generalised notions of rationality, commutativity and associativity as natural consequences of the definition. Finally, we show that for a certain choice of vertex group, axiomatic vertex algebras correspond bijectively to algebras in the relaxed multilinear category of representations of a vertex group.

Contents

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1 Introduction

1.1 Motivation

The theory of (non-supersymmetric) axiomatic vertex algebras has as its data, a complex vector space, V, together with a **vertex operator**:

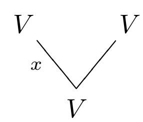
$$Y(\cdot,x)\cdot : V\otimes V \longrightarrow V[\![x]\!][x^{-1}],$$

as well as a distinguished vector $|0\rangle$ and an automorphism $T : V \to V$. So for vectors $a, b \in V$, the vertex operator can be written as

$$Y(a,x)b = x^{-k}\sum_{i\geq 0} c_i x^i,$$

where $c_i \in V$ and $k \geq 0$. We would like to describe carefully the products of vertex operators, and to give a precise description of what types of maps arise from such products.

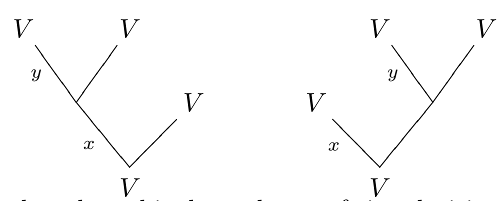
We represent the collection of all such vertex operators by the labelled tree



By extending the domain of the vertex operator pointwise to $V[\![x]\!][x^{-1}]$, we have two ways to take the product of a pair of vertex operators, giving maps:

$$Y(\cdot,x)Y(\cdot,y)\cdot : \quad V\otimes V\otimes V \longrightarrow \quad V[\![x]\!][x^{-1}][\![y]\!][y^{-1}] \tag{1.1}$$
$$Y(Y(\cdot,y)\cdot,x)\cdot : \quad V\otimes V\otimes V \longrightarrow \quad V[\![x]\!][x^{-1}][\![y]\!][y^{-1}]. \tag{1.2}$$

Immediately we notice that the collection of all maps from $V^{\otimes 3}$ to $V[\![x]\!][x^{-1}][\![y]\!][y^{-1}]$ contains many maps which can not be realised as the composite of two vertex operators. For example, the singularity in the variable y only depends on two of the copies of V in $V^{\otimes 3}$. Composing two copies of the labelled tree, we can represent the collection of all maps arising as the composite of two vertex operators by



This labeled tree notation makes clear this dependency of singularities on inputs. Looking more closely at the product, $Y(\cdot,x)Y(\cdot,y)\cdot$, we see that a more accurate description of the space of maps defined by this product is given by the collection,

$$\mathrm{Hom}\Big(V\otimes V, \mathrm{Hom}(V, V[\![x]\!][x^{-1}])[\![y]\!][y^{-1}]\Big).$$

We can repeat this process, so that the collection of n-fold products of vertex operators, $Y(\cdot,x_n)\cdots Y(\cdot,x_1)\cdot$ can be described exactly as the product,

$$\mathrm{Hom}\Bigg(V\otimes V, \mathrm{Hom}\Big(V,\cdots \mathrm{Hom}(V, V[\![x_n]\!][x_n^{-1}])\cdots\Big)[\![x_1]\!][x_1^{-1}]\Bigg).$$

Similarly, this space describes the other type of n-fold product of vertex operators,

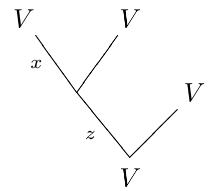
$$Y\Big(Y\big(\cdots Y(Y(\cdot,x_1)\cdot,x_2)\cdots\big)\cdot,x_n\Big)\cdot\,.$$

The difficulty arises when we begin to consider the space of products of vertex operators which contain both types of composition. Take for example the composite,

$$Y(Y(\cdot,x)\cdot,z)Y(\cdot,y)\cdot\,.$$

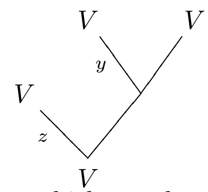
This product can be achieved either by composing three vertex operators, $Y(\cdot,x)\cdot$, $Y(\cdot,y)\cdot$, and $Y(\cdot,z)\cdot$, or by appropriately composing a vertex operator $Y(\cdot,y)\cdot$ with an element of

$$\mathrm{Hom}(V\otimes V,\mathrm{Hom}(V,V[\![z]\!][z^{-1}])[\![x]\!][x^{-1}]) =$$

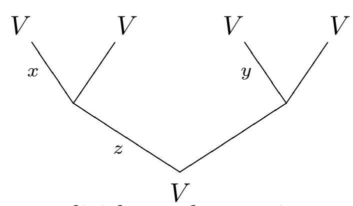


or even by composing $Y(\cdot,x)\cdot$ in the other way with an element of

$$\mathrm{Hom}(V\otimes V,\mathrm{Hom}(V,V[\![z]\!][z^{-1}])[\![y]\!][y^{-1}]) =$$



Either way, we would like to understand the space of all maps which can be realized as any of the composites given above. Using the tree notation, we shall denote the space of all such maps by



but this space can not be described as explicitly as the previous spaces of products of vertex operators.

In addition to the desire to simply describe these spaces of products of vertex operators, we would like to be able to relate them in such a way as to take account of the axioms of the vertex algebra. The usual way of working with products of vertex operators is to consider them inside very large spaces of formal distributions. While this has the benefit of leaving plenty of room to manipulate power series, it tends to obscure the important features of the vertex operators. In this paper we shall take the opposite tack, opting instead to use this minimal description of the spaces in which these products of vertex operators live. In this way, we will be able to make clear the essential features of composition of vertex operators.

1.2 Outline

The idea behind this paper is the extraction the essential features of axiomatic vertex algebras, with the ultimate goal of showing how they can be seen to arise naturally in a certain context. By demonstrating this naturality, the generalisation of vertex algebras to more general settings becomes simply a matter of choosing the correct categories in which to work, with the long term goal of applying them to higher dimensional field theories.

We shall approach the problem of giving an abstract account of vertex algebras by first considering carefully the vertex operator. After reviewing the definition of a vertex algebra, we define a vertex group which ties together the power series and the infinitesimal translation operator. Temporarily setting aside the singularities of a vertex operator, we show how holomorphic vertex algebras arise naturally when considering representations of the vertex group.

Next we look at the role of singularities in the definition of a vertex operator. We show how to use the elementary vertex structure of our vertex group to define a space of singular functions, and after considering composition of these singular functions we define spaces of singular functions parameterised by binary trees. Our definitions, while motivated by the Borcherds definition in [3], are different because they emphasise the relation of singularities on inputs. For the classical vertex group, we demonstrate the correspondence between a binary singular functions and vertex operators.

In the next section we consider the axioms for our vertex algebra. After discussing the types of multi maps which arise for one leafed trees, we see that the vacuum axioms lead us to consider maps associated to trees with zero leaves. We show how the locality condition for vertex operators suggests the notion of singular maps associated to a three leafed tree with no internal vertices. We finish this section by defining singular maps associated to an arbitrary tree, and that natural maps between n leafed trees give rise to natural maps between corresponding singular functions.

We finish this paper by describing the general categorical structure satisfied by these singular maps. They form a relaxed multilinear category, and we show that a vertex algebra is just an algebra in this category.

For clarity, our presentation does not give the supersymmetric details, but the generalisation can be made easily.

1.3 Notation

Throughout this paper we will often refer to modules of polynomials, power series, and Laurent series. Given any R-module, B (where R is a commutative ring with 1), our convention will be to denote the collection of power series with coefficients in B by $B[\![x]\!]$, and the collection of Laurent series by $B[\![x]\!][x^{-1}]$. The notation can be combined to form larger modules such as $B[\![x]\!][x^{-1}][\![y]\!]$, the collection of power series in the variable y, whose coefficients are Laurent series in the variable x. Note that $B[\![x]\!][x^{-1}][\![y]\!] \supsetneq B[\![x, y]\!][x^{-1}]$ because, for example, $\sum_{i\geq 0} x^{-i}y^i$ is contained in the first module but not the second.

We also adopt Sweedler's notation [17] to denote comultiplication by $\Delta(h) = \sum_{(h)} h_{(1)}\otimes h_{(2)}$ for any coalgebra element h.

Acknowledgements

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2 Classical Vertex Group and its Representations

In the introduction we concentrated on examining the products of vertex operators. We purposely avoided the question of what types of homomorphisms we were considering, because we were mainly concerned with demonstrating the dependence of singularities on products of vertex operators. In this section we return to the vertex operator itself, examining it closely and paying special attention to its interaction with the infinitesimal translation operator.

Before beginning we recall the definition of a vertex algebra (following the definition in [11]).

Definition 2.1. *A **vertex algebra** consists of a complex vector space V (the **state space**) together with an endomorphism, $T : V \to V$ (the **infinitesimal translation operator**), a distinguished vector denoted $|0\rangle \in V$ (the **vacuum vector**), and a linear map (the **vertex operator**):*

$$Y(\cdot, x)\cdot : V\otimes V \to V[\![x]\!][x^{-1}],$$

taking any $a\otimes b \in V\otimes V$ to $Y(a, x)b \in V[\![x]\!]$. The axioms for a vertex algebra say that

Vacuum axioms: *For any $a \in V$, the vacuum satisfies:*

$$T|0\rangle = 0 \tag{2.1}$$
$$Y(|0\rangle, x)a = a \tag{2.2}$$
$$Y(a, x)|0\rangle|_{x=0} = a. \tag{2.3}$$

Translation covariance axiom: *The infinitesimal translation operator interacts with a vertex operator according to:*

$$[T, Y(a,x)] = \partial_x Y(a,x) : V \to V[\![x]\!][x^{-1}]. \tag{2.4}$$

Locality axiom: *And, for any $a, b \in V$, there exists some $N \gg 0$ such that the following holds:*

$$(x-y)^N [Y(a,x), Y(b,y)] = 0. \tag{2.5}$$

If we consider the free group ring generated by the infinitesimal translation operator, denoted G, then V is a G-module simply by virtue of the fact that T is an endomorphism of V. From a simple vertex algebra calculation, one can show that $\partial_x Y(a,x) = Y(Ta,x)$ (see [11, Prop. 4.8]). So, from the translation covariance axiom, we see that given any $b \in V$, we have an equality of Laurent series,

$$Y(a,x)Tb + Y(Ta,x)b = TY(a,x)b.$$

Now, considering the vertex operator as a map from $V \otimes V$ to $V[\![x]\!][x^{-1}]$, this presentation of the translation covariance axiom just says that a vertex operator is a G-invariant map, where G acts on $V \otimes V$ by

$$\begin{array}{rcl} G \otimes (V \otimes V) & \longrightarrow & V \otimes V \\ T \otimes a \otimes b & \longmapsto & Ta \otimes b + a \otimes Tb. \end{array}$$

From this account, it is clear that a general vertex algebra is going to be some type of module. In the next section we will define a vertex group, which will be the algebra over which we will want to work. We show that G is a nontrivial example of a vertex group. We then move on to consider modules over a vertex group, and modules over G in particular. We finish this section by considering ways of expressing maps between G-modules and we show how holomorphic vertex algebras arise naturally from this presentation.

2.1 Definition of a Vertex Group

This section reviews some important definitions from [3]. In particular, we define an elementary vertex structure on a cocommutative Hopf algebra which we use to introduce the notion of a vertex group as a Hopf algebra together with a specific choice of elementary vertex structure.

Recall that a Hopf algebra is a module H over a commutative ring R (with unit) that has both the structure of an algebra and a coalgebra. It also possesses an antipode map $S : H \to H$ which is both a map of algebras and a map of coalgebras, and serves to connect the algebra and coalgebra structures.

Definition 2.2. *Let H be a Hopf algebra over a commutative ring R, and let $H^* = \mathrm{Hom}_R(H, R)$ be the collection of R-linear maps from H to R. We say that an R-module, K, gives an **elementary vertex structure** on H if it is an associative algebra over the algebra H^* satisfying the following properties:*

1. ***(Closure under left and right translation)*** *K is a 2-sided H-module such that the product on K is invariant under the left and right actions of H. By this we mean that for $h \in H$ and $k, l \in K$, we have:*

   $$h \cdot (kl) = \sum_{(h)} (h_{(1)} \cdot k)(h_{(2)} \cdot l) \qquad\qquad (kl) \cdot h = \sum_{(h)} (k \cdot h_{(1)})(l \cdot h_{(2)}).$$

   We also require that the unit map for the algebra, $\eta : H^* \to K$, be a homomorphism of 2-sided H-modules.

2. ***(Closure under Inversion)*** *The antipode on H gives rise to a map S^* on H^* that can be extended to an R-linear map on K. By abuse of notation, we will refer to this map as $S : K \to K$, and because we are extending the the dual map it is not quite an H^*-algebra map, but instead satisfies $S(kl) = S(l)S(k)$ and $S(1) = 1$ for $k, l, 1 \in K$.*

Definition 2.3. *If H is a cocommutative Hopf algebra and K is an elementary vertex structure on H which is commutative and satisfies $S^2 = 1_K$, then we say that we have an* ***elementary vertex group****, G. H will be referred to as the* ***group ring*** *of the vertex group G or the* ***underlying Hopf algebra****, and K will be called the* ***ring of singular functions on*** *G.*

Note. *Since there are examples of vertex groups that are more general than this definition allows, we have chosen the name elementary vertex groups. Throughout the rest of this paper we will only be working with elementary vertex groups, so for simplicity, we will refer to them simply as vertex groups.*

Definition 2.4. *For any H-module, B, we abuse terminology by calling $\mathrm{Hom}_R(H^n, B)$ the* ***collection of nonsingular functions from*** *$G^{\otimes n}$* ***to*** *B. This will also be written $\mathrm{Hom}_R(G^{\otimes n}, B)$.*

Notice that for any cocommutative Hopf algebra, H, letting $K = H^*$ gives H the structure of a trivial vertex group. This is a 2-sided H-module with left and right actions given as above for $g, h \in H$ and $f \in H^*$,

$$(h \cdot f)(g) = \sum_{(h)} h_{(1)} f(S(h_{(2)})g) \qquad\qquad (f \cdot h)(g) = f(hg). \tag{2.6}$$

Since we are working with the dual of a Hopf algebra this is closed under left and right translation and under inversion.

2.2 Classical Vertex Group

The Hopf algebra which will prove most important for our later work is the complex polynomial algebra in one variable, $H = \mathbb{C}[T]$. For simplicity, we shall denote powers of T by $D^{(i)} = T^i/i!$. Thus we have an associative algebra with multiplication $\mu(D^{(i)} \otimes D^{(j)}) = \binom{i+j}{i} D^{(i+j)}$ for any $i, j \geq 0$ and unit $D^{(0)}$. We shall denote multiplication as $D^{(i)} \cdot D^{(j)}$.

This module has the additional structure of a Hopf algebra from the following three maps:

$$\text{Coalgebra Structure:} \qquad \begin{aligned} \Delta : H &\to H \otimes H \\ D^{(i)} &\mapsto \sum_{p+q=i} D^{(p)} \otimes D^{(q)} \end{aligned} \qquad\qquad \begin{aligned} \epsilon : H &\to \mathbb{C} \\ D^{(i)} &\mapsto \begin{cases} 1 & i = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\text{Antipode:} \qquad \begin{aligned} S : H &\to H. \\ D^{(i)} &\mapsto (-1)^i D^{(i)}. \end{aligned}$$

Note that in particular this structure is cocommutative (i.e., $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$).

It is easy to see that the linear dual of H, $H^* = \mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C})$, is the ring of power series in one variable, $H^* = \mathbb{C}[\![x]\!]$. The dual pairing is given by letting $D^{(1)}$ act as differentiation on the x variable evaluated at $x = 0$. We can extend this action of H on its dual to give an H-module structure on H^* by defining the obvious action of derivation $D^{(j)} \cdot x^i = \binom{i}{j} x^{i-j}$, so that H is just a group ring of derivations on $\mathbb{C}[\![x]\!]$. It is easy to see that more generally, $\mathrm{Hom}_{\mathbb{C}}(H^{\otimes n}, \mathbb{C})$ can be identified with $\mathbb{C}[\![x_1, \ldots, x_n]\!]$, where a map f is paired with $\sum r_{i_1,\ldots,i_n} x_1^{i_1}, \ldots, x_n^{i_n}$ when $f(D^{(i_1)} \otimes \ldots \otimes D^{(i_n)}) = r_{i_1,\ldots,i_n}$.

If we choose our elementary vertex structure to be $K = \mathbb{C}[\![x]\!][x^{-1}]$, the space of Laurent series in one variable, we see immediately that it is an associative algebra over H^* by construction. In other words, it is an infinite dimensional vector space over H^* spanned by $\{x^{-i} | i \geq 0\}$, with multiplication and unit maps giving it the structure of an algebra. The action of H is the obvious extension of the action of derivation on H^*. So for any $j \geq 0$ and $i \in \mathbb{Z}$ we have $D^{(j)} \cdot x^i = \binom{i}{j} x^{i-j}$. Recall that for negative i, the binomial coefficient is given by

$$\binom{i}{j} = \frac{i(i-1)\cdots(i-j+1)}{j(j-1)\cdots 1} = (-1)^j \binom{j-i-1}{j}.$$

If $i = j$ then this gives back our dual pairing between x^i and $D^{(i)}$. We extend the antipode to K by defining $S(x^i) = (-1)^i x^i$ for all $i \in \mathbb{Z}$. By the product rule for derivation, we also see that the action of

H on the product of two elements in K commutes with the multiplication by using the diagonal map. In other words, for any $j \geq 0$, $i_1, i_2 \in \mathbb{Z}$;

$$\begin{aligned} D^{(j)} \cdot (x^{i_1} x^{i_2}) &= \mu\big(\Delta(D^{(j)}) \cdot (x^{i_1} \otimes x^{i_2})\big) \\ &= \sum_{p+q=j} \mu\big((D^{(p)} \cdot x^{i_1}) \otimes (D^{(q)} \cdot x^{i_2})\big) \\ &= \sum_{p+q=j} \binom{i_1}{p} \binom{i_2}{q} x^{i_1+i_2-p-q} \\ &= \binom{i_1+i_2}{j} x^{i_1+i_2-j}. \end{aligned}$$

and so this vertex algebra is closed under left and right translation. We shall call this the classical vertex group since it will give rise to the vertex algebras of [11]. It shall be denoted G and will be the most important example for the sections that follow.

2.3 Representations of the Classical Vertex Group

We now return our attention to the case of a general vertex group, G, with underlying Hopf algebra, H. An R-module, B, is a **representation** of the vertex group if it is a representation of the the underlying Hopf algebra. Hence the category of modules for a vertex group, G, is exactly the category of H-modules. When we refer to the action of G on a module, we mean that its group ring is acting on the module. The reason for introducing the category of G-modules is because we will use the elementary vertex structure of the vertex group to provide this category with some additional singular structure. This additional structure will make it into a relaxed multilinear category (see definition 5.1 below). Before we introduce this additional structure we look more closely at the underlying category of modules for a vertex group.

Note. *We will want to consider the collection of all representations of a vertex group. We shall not restrict our attention to the full subcategory of finite dimensional representations because we want to allow ourselves the freedom to work with representations possessing a singular structure freely generated as an algebra.*

We begin by pointing out that since the underlying Hopf algebra for any vertex group is cocommutative, the category of G-modules possesses a symmetric tensor product, which is just the tensor product of objects considered as R-modules. Given G-modules A and B, the action of G on $A \otimes B$ is given by the diagonal map. So we can think of $A \otimes B$ as having both a G-action and a $(G \otimes G)$-action. More generally, given the tensor product of a collection of G-modules, $A_1 \otimes \cdots \otimes A_n$, for every $1 \leq i \leq n$ there exist actions of $G^{\otimes i}$ corresponding to the different ways of comultiplying $G^{\otimes i} \to G^{\otimes n}$.

Next we notice that the collection of R-linear maps between any two G-modules A and B, denoted $\mathrm{Hom}_R(A, B)$, can be given the structure of a module in a number of ways. Given any such map, $f : A \to B$, we first define an action of G on f for any $a \in A$ and $g \in G$ by

$$(g \cdot f)(a) = \sum_{(g)} g_{(1)} \cdot f(S(g_{(2)}) \cdot a). \tag{2.7}$$

We say that f is **invariant** under the action of G if $g \cdot f = \epsilon(g) f$ for all $g \in G$. This is equivalent to saying $f(g \cdot a) = g \cdot f(a)$, or in other words the G-invariant ring maps are exactly the G-module morphisms.

There also exist two additional actions of G on $\mathrm{Hom}_R(A, B)$; one where G acts on the domain, the other where it acts on the codomain. The G-action on the domain of a map $f : A \to B$ is a right action on f, while the G-action on the codomain of f is a left action. In fact, considering $\mathrm{Hom}_R(A_1 \otimes \ldots \otimes A_n, B)$ we see that in addition to being a map of G-modules, the domain possesses the additional structure of a module for $G^{\otimes n}$. So $\mathrm{Hom}_R(A_1 \otimes \ldots \otimes A_n, B)$ possesses the structure of a (right) $G^{\otimes n}$-module by action on the domain of f (and similarly, a (left) G-module for action on the codomain, B). The fact that $\mathrm{Hom}_R(A_1 \otimes \ldots \otimes A_n, B)$ can be a module for both G and $G^{\otimes n}$ will be important when we look at the composition of singular maps below (see section 3.2).

Returning to the particular case of the classical vertex group, we have shown that $\mathrm{Hom}_{\mathbb{C}}(G, \mathbb{C}) \cong \mathbb{C}[\![x]\!]$. We see immediately that for any G-module, B, the linear dual, $\mathrm{Hom}_{\mathbb{C}}(G, B)$, is isomorphic as a G-module

to $B[\![x]\!]$, where $B[\![x]\!]$ denotes the collection of power series in x with coefficients in B. This isomorphism between maps linear maps and power series can be extended to maps from n-fold products of G for any $n \geq 0$. In such a case we have $\mathrm{Hom}_{\mathbb{C}}(G^{\otimes n}, B) \cong B[\![x_1, \dots, x_n]\!]$ (where we have tacitly included the case where $n = 0$ and $\mathrm{Hom}_{\mathbb{C}}(G^{\otimes 0}, B) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, B) \cong B$).

From the discussion above we know that $G^{\otimes n}$ has the structure of a G-module using the n-fold diagonal map, so the collection of linear maps $\mathrm{Hom}_{\mathbb{C}}(G^{\otimes n}, B)$ also has the structure of a G-module as described above. Under the isomorphism $\mathrm{Hom}_{\mathbb{C}}(G^{\otimes n}, B) \cong B[\![x_1, \dots, x_n]\!]$, the action of $D^{(i)} \in G$ on any $b(x_1, \dots, x_n) \in B[\![x_1, \dots, x_n]\!]$ is given by:

$$D^{(i)} \cdot b(x_1, \dots, x_n) = \sum_{j_0 + \dots + j_n = i} (-1)^{j_0} \big(D_B^{(j_0)} + D_{x_1}^{(j_1)} \cdots + D_{x_n}^{(j_n)}\big) b(x_1, \dots, x_n). \tag{2.8}$$

In the previous expression, D_B denotes the action of G on the coefficients of the power series $b(x_1, \dots, x_n)$, and where $D_{x_k}^{(j_k)}$ denotes the action of differentiation on the variable x_k. As before, we say that $b(x_1, \dots, x_n)$ is invariant under the action of H (or G) if $D^{(i)} \cdot b(x_1, \dots, x_n)$ is zero. This is the same as requiring that $b(x_1, \dots, x_n)$ satisfy the following for all $i \geq 0$:

$$D_B^{(i)} b(x_1, \dots, x_n) = \sum_{j_1 + \dots + j_n = i} \big(D_{x_1}^{(j_1)} + \dots + D_{x_n}^{(j_n)}\big) b(x_1, \dots, x_n). \tag{2.9}$$

The collection of all such invariant elements is denoted $\mathrm{Hom}_G(G^{\otimes n}, B)$.

Working out equation (2.8) explicitly for the case of $\mathrm{Hom}_{\mathbb{C}}(G^{\otimes 2}, B) \cong B[\![x, y]\!]$, we have

$$\begin{aligned} D^{(1)} \cdot (\sum_{i,j \geq 0} b_{i,j} x^i y^j) &= \sum_{i,j \geq 0} (D^{(1)} \cdot b_{i,j}) x^i y^j - (D_x^{(1)} + D_y^{(1)}) \sum_{i,j \geq 0} b_{i,j} x^i y^j \\ &= \sum_{i,j \geq 0} (D^{(1)} b_{i,j}) x^i y^j - \sum_{i,j \geq 0} i b_{i,j} x^{i-1} y^j - \sum_{i,j \geq 0} j b_{i,j} x^i y^{j-1}. \end{aligned}$$

And from equation (2.9), we see that the G-invariant elements of $B[\![x, y]\!]$ are exactly those which satisfy $D^{(1)} \cdot b_{i,j} = (i+1) b_{i+1,j} + (j+1) b_{i,j+1}$, or more generally

$$D^{(k)} \cdot b_{i,j} = \sum_{p+q=k} \binom{p+i}{i} \binom{q+j}{j} b_{p+i,q+j}. \tag{2.10}$$

For any n, this equation also has an "inverted form" in which we can write any $b_{i_1, \dots, i_n}$ as a sum of $b_{j_1, \dots, j_n}$ where one of the indices $(j_1, \dots, j_n)$ is set to zero. For $n = 2$ we can write $b_{i,j}$ as:

$$b_{i,j} = \sum_{p+q=j} (-1)^p \binom{i+p}{i}^{D^{(q)}} b_{i+p,0} \tag{2.11}$$

$$= \sum_{p+q=i} (-1)^q \binom{j+q}{j}^{D^{(p)}} b_{0,j+q}. \tag{2.12}$$

This ability to represent G-invariant power series in various way will be important for encapsulating duality for vertex algebras.

2.4 Extended Representation of Morphisms - Holomorphic Vertex Algebras

Now that we have a characterisation of modules for a vertex group, we shall look more closely at maps between them. We would like to describe module maps in a way which will allow us to naturally add the elementary vertex structure. This treatment will avoid a very general abstract treatment, instead aiming to provide the flavour of theory. The details can be found in [15].

Given any linear map between G-modules, $f : A \to B$, we define a unique G-linear map $\widehat{f} : A \to \mathrm{Hom}_R(G, B)$ by:

$$\widehat{f}(a)(g) = f(g \cdot a).$$

This pairing between linear maps and G-linear maps is actually an isomorphism in the category of G-modules, so we may easily pass from one description to the other. We shall call this an **extended representation** of f. If the map f was also G-invariant, then the extended representation of f would be a G-linear map from A to G-invariant elements of $\mathrm{Hom}_R(G, B)$. These extended representations can also be used to reexpress linear maps of the form $f : A_1 \otimes \cdots \otimes A_n \to B$ as $G^{\otimes n}$-module maps $\widehat{f} : A_1 \otimes \cdots \otimes A_n \to \mathrm{Hom}_R(G^{\otimes n}, B)$, where the action of $G^{\otimes n}$ on the domain, $A_1 \otimes \cdots \otimes A_n$, passes to an action on the domain of the maps $\mathrm{Hom}_R(G^{\otimes n}, B)$.

Example 2.5. In the case of the classical vertex group, this means that linear maps between G-modules $f : A_1 \otimes \cdots \otimes A_n \to B$, are the same as $G^{\otimes n}$-linear maps

$$\widehat{f} : A_1 \otimes \cdots \otimes A_n \to B[\![x_1, \ldots, x_n]\!].$$

In particular, we have that the action of G on any of the A_i carries over to differentiation of x_i as

$$\widehat{f}(a_1 \otimes \cdots \otimes D^{(1)} a_i \otimes \cdots \otimes a_n) = \partial_{x_i} \widehat{f}(a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_n).$$

The composition of maps in this extended representation is slightly delicate. Given ordinary linear maps of G-modules, say $f : A \to B$ and $g : B \to C$, then clearly these compose to give a map $g \circ f : A \to C$. Now f and g can be considered in their extended representation, i.e., G-linear maps $\widehat{f} : A \to \mathrm{Hom}_R(G, B)$ and $\widehat{g} : B \to \mathrm{Hom}_R(G, C)$, which compose to give a G-linear map $\widehat{g} \circ \widehat{f} : A \to \mathrm{Hom}_R(G^{\otimes 2}, C)$. But the ordinary composite $g \circ f$ has an extended representation $\widehat{g \circ f} : A \to \mathrm{Hom}_R(G, C)$. We would expect these to be related, but in fact, we need to require that f be G-invariant in order to relate/reduce the extra factor of G in the codomain of $\widehat{g} \circ \widehat{f}$. Intuitively, this problem appears because the action of G on B appears in $\mathrm{Hom}_R(G, C)$, but the composite $g \circ f$ "hides" that action. So by taking f to be G -invariant we eliminate the action of G on f explicitly.

More generally, maps in the extended representation compose pointwise as multilinear functions in the usual way, provided that all maps (except possibly the bottom map) are G-invariant.

Example 2.6. Again we consider the case of the classical vertex group. Let A_i, B_j, C_l and D be G-modules. Given a $G^{\otimes n}$-linear map $g : C_1 \otimes \cdots \otimes C_n \to D[\![z_1, \ldots, z_n]\!]$, and G-invariant maps $f : A_1 \otimes \cdots \otimes A_p \to C_1[\![x_1, \ldots, x_p]\!]$, $k : B_1 \otimes \cdots \otimes B_q \to C_2[\![y_1, \ldots, y_q]\!]$ ($G^{\otimes p}$-linear and $G^{\otimes q}$-linear respectively), then the composite is a $G^{\otimes (p+q+n-2)}$-linear map

$$g \circ (f \otimes k) : A_1 \otimes \cdots \otimes A_p \otimes B_1 \otimes \cdots \otimes B_q \otimes C_3 \otimes \cdots \otimes C_n \to D[\![x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_n]\!]$$

which satisfies $\partial_{z_1} = \partial_{x_1} + \cdots + \partial_{x_p}$ and $\partial_{z_2} = \partial_{y_1} + \cdots + \partial_{y_q}$. In other words, the map $g \circ (f \otimes k)$ factors through $D[\![x_1 + z_1, \ldots, x_p + z_1, y_1 + z_2, \ldots, y_q + z_2, z_3, \ldots, z_n]\!]$. Also, it makes sense to consider compositions in a particular order, say $(g \circ k) \circ f$ or $(g \circ f) \circ k$, and these are equal.

Claim 2.7. *Let* ***H-Mod*** *be the category of representations of the underlying Hopf algebra, H, of the classical vertex group. Then category of holomorphic vertex algebra is isomorphic to the category of H-invariant commutative algebras in* ***H-Mod***.

Proof. In order to make sense of this statement, we will first review both the definition of a holomorphic vertex algebra, and the definition of an algebra in a category.

Definition 2.8. *A* ***holomorphic vertex algebra*** *is a vertex algebra without singularities. In other words, the vertex operator of a holomorphic vertex algebra is a map:*

$$Y(\cdot, x)\cdot : V \otimes V \to V[\![x]\!].$$

It follows that the locality axiom reduces to the statement that products of vertex operators commute.

The collection of holomorphic vertex algebras is given the structure of a category by defining a morphism in the category of holomorphic vertex algebras to be a map of complex vector spaces taking vacuum to vacuum, commuting with multiplication, and respecting the actions of the infinitesimal translation operators.

Definition 2.9. *A **commutative (associative) algebra** in a symmetric tensor category consists of an object A, a multiplication map $\mu : A \otimes A \to A$ which is invariant under the symmetry action for the tensor product, and a unit for the multiplication $\eta : I \to A$ satisfying the usual axioms for associativity and unit (where I is the unit for the tensor product).*

For the category of representations of the Hopf algebra H, the unit for multiplication is $\mathbb{C}$. and the multiplication map μ is H-invariant. Given any such algebra in this category of representations, where the multiplication map μ is H-invariant, we form a holomorphic vertex algebra as follows. We begin by taking $T = D^{(1)}$ as the infinitesimal translation operator, and take the vacuum vector to be $\eta(1)$. Then by considering μ in the extended representation, we have a map

$$\widehat{\mu} : A \otimes A \to A[\![x, y]\!],$$

which satisfies

$$\begin{aligned} \widehat{\mu}(Ta \otimes b) &= \partial_x \widehat{\mu}(a \otimes b) && (2.13) \\ \widehat{\mu}(a \otimes Tb) &= \partial_y \widehat{\mu}(a \otimes b) && (2.14) \\ T\widehat{\mu}(a \otimes b) &= \widehat{\mu}(Ta \otimes b) + \widehat{\mu}(a \otimes Tb). && (2.15) \end{aligned}$$

From this we define a vertex operator on A to be

$$Y(\cdot, x)\cdot = \widehat{\mu}(\cdot \otimes \cdot)|_{y=0} : A \otimes A \to A[\![x]\!]. \tag{2.16}$$

Checking that this satisfies the axioms for a holomorphic vertex algebra, we see immediately that the vacuum axioms are satisfied because H acts trivially on $\mathbb{C}$, and because $\mu(\eta(1) \otimes a) = a$ for all $a \in V$. Writing out the translation covariance axiom, we use the H-invariance of $\widehat{\mu}$ to give:

$$\begin{aligned} T\widehat{\mu}(a \otimes \cdot)|_{y=0} - \widehat{\mu}(a \otimes T\cdot)|_{y=0} &= \widehat{\mu}(Ta \otimes \cdot)|_{y=0} + \widehat{\mu}(a \otimes T\cdot)|_{y=0} - \widehat{\mu}(a \otimes T\cdot)|_{y=0} && (2.17) \\ &= \widehat{\mu}(Ta \otimes \cdot)|_{y=0} && (2.18) \\ &= \partial_x \widehat{\mu}(a \otimes \cdot)|_{y=0}. && (2.19) \end{aligned}$$

And the locality axiom follows from the commutativity of $\widehat{\mu}$. Notice that $\widehat{\mu}(a \otimes b) = Y(a, x)Y(b, y)|0\rangle$.

Similarly, given any vertex algebra, we can easily define an algebra in the category of of representations of the Hopf algebra H by taking $\eta(r) = r|0\rangle \in V$ for any $r \in R$, and $\mu(\cdot \otimes \cdot) = Y(\cdot, x) \cdot |_{x=0}$. The unit axioms follow from the properties of the vacuum, and associativity follows from locality. The translation covariance axiom says that the multiplication is H-invariant. And, locality axiom acting on the vacuum says that this multiplication is commutative.

If $(V, Y(\cdot, x)\cdot, T, |0\rangle)$ is a vertex algebra, and the corresponding H-invariant algebra is (V, μ, η), then from this algebra, we get back the vertex algebra, $(V, Y'(\cdot, z)\cdot, T, |0\rangle)$. If $Y(a, x)b = \sum_{i \geq 0} c_i x^i$ for $a \otimes b \in V \otimes V$, then

$$\widehat{\mu}(a \otimes b) = \sum_{i,j \geq 0} \left[Y(D^{(i)}a, x) D^{(j)} b \right]_{x=0} z^i y^j$$

so setting $y = 0$, we have

$$\begin{aligned} \widehat{\mu}(a \otimes b)|_{y=0} &= \sum_{i \geq 0} \left[Y(D^{(i)}a, x) b \right]_{x=0} z^i \\ &= \sum_{i \geq 0} \left[\sum_{j \geq 0} \binom{j+i}{i} c_{j+i} x^j \right]_{x=0} z^i \\ &= \sum_{i \geq 0} c_i z^i, \end{aligned}$$

and so $Y = Y'$. Similarly, starting with an H-invariant algebra, (A, μ, η), the process of creating a holomorphic vertex algebra and then mapping back to an H-invariant algebra clearly maps (A, μ, η) to itself.

Finally, a morphism in the category of holomorphic can be seen to correspond exactly to a morphism of algebras in the category of H-modules. Because this pairing of objects in each category is completely natural we have an isomorphism of categories and so our claim is proved. □

3 Singularities

Up to this point, we have concentrated on the underlying Hopf algebra structure of our vertex group. We now use the elementary vertex structure of the vertex group to add singularities to the extended representations of morphisms of G-modules, paying special attention to the case for the classical vertex group.

It is in this chapter that this paper departs from Richard Borcherds' paper [3]. The ideas in this section were inspired by that paper, but the collections of singular maps we shall define are much smaller than those defined in his paper. In fact, they will be defined in such a way as to make their relationship to his singular maps clear.

3.1 Localisation

The following definitions illustrate how we will put together the space of nonsingular functions from $G^{\otimes n}$ to B with the ring of singular functions on G in order to arrive at a notion of singular functions from $G^{\otimes n}$ to B. These will also be referred to as *singular functions of type K*.

Definition 3.1. *For $1 \leq i < j \leq n$ and any (right) $\mathrm{Hom}_R(G^{\otimes n}, R) = (G^{\otimes n})^*$-module, M, the* ***localisation of*** *M* ***at*** *(i,j) is defined to be $M \otimes_{G^*} K$, where M is given the structure of an G^*-module by the R-module dual of the map*

$$\begin{array}{rccc} \widehat{f}_{ij}: & G^{\otimes n} & \longrightarrow & G. \\ & g_1 \otimes \cdots \otimes g_n & \mapsto & g_i S(g_j) \end{array} \tag{3.1}$$

We have already seen that the space of nonsingular functions from $G^{\otimes n}$ to B has the structure of both an G-module and an $G^{\otimes n}$-module. Because of the coassociativity of G, it can also be considered as an $(G^{\otimes n})^*$-module where the action of any $\alpha \in (G^{\otimes n})^*$ on $f : G^{\otimes n} \to B$ is

$$(\alpha \cdot f)(h_1 \otimes \cdots \otimes h_n) = \alpha(h_1 \otimes \cdots \otimes h_n) \cdot f(h_1 \otimes \cdots \otimes h_n) \tag{3.2}$$

for all $h_1 \otimes \cdots \otimes h_n \in G^{\otimes n}$. With this action defined, it makes sense to localise the space of nonsingular functions from $G^{\otimes n}$ to B.

Lemma 3.2. *The localisation of $\mathrm{Hom}_R(G^{\otimes n}, B)$ is also a $(G^{\otimes n})^*$-module, and sequential localisations commute.*

Proof. This is clear from the action defined in equation (3.2) and the coassociativity of G. ☐

Definition 3.3. *The* ***space of singular functions*** *from $G^{\otimes n}$ to B, denoted $Fun(G^{\otimes n}, B)$, is defined to be the localisation of the space of nonsingular functions from $G^{\otimes n}$ to B at all (i,j) for $1 \leq i < j \leq n$. It is the space*

$$\mathrm{Hom}_R(G^{\otimes n}, B) \bigotimes_{\substack{f_{i,j} \\ 1 \leq i < j \leq n}} K$$

The space of singular functions from $G^{\otimes n}$ to B is a module for a number of different actions. Firstly, from lemma 3.2, we have that $Fun(G^{\otimes n}, B)$ is a $(G^{\otimes n})^*$-module. Next we know that there is a left action of G on B which carries over to $Fun(G^{\otimes n}, B)$. We also know that there is a (left) diagonal action of G on the domain of the nonsingular maps $\mathrm{Hom}_R(G^{\otimes n}, B)$. An easy check shows that this diagonal action carries over to a trivial action on K via the maps $f_{i,j}$ defined in equation (3.1). Hence the standard action of G on any singular function is the same as the standard action of G on the nonsingular functions given by (2.7). This gives us the following lemma:

Lemma 3.4. *If we define the* ***space of singular G-invariant functions*** *from $G^{\otimes n}$ to B, denoted $Fun_G(G^{\otimes n}, B)$, to be the localisation of $\mathrm{Hom}_G(G^{\otimes n}, B)$ at all (i,j) for $1 \leq i < j \leq n$, then this coincides with the G-invariant elements of $Fun(G^{\otimes n}, B)$ under the standard action of G.*

Finally we know that the nonsingular maps, $\mathrm{Hom}(G^{\otimes n}, B)$, possesses a (right) action of $G^{\otimes n}$ on their domain. This action can be extended to singular maps by using comultiplication to act on each of the $\otimes_{f_{i,j}} K$ terms through that map $f_{i,j}$.

Example 3.5. If we are working with a trivial vertex group (i.e., $K = H^*$) then the localisation of the nonsingular functions from $G^{\otimes n}$ to B has no effect, and we just have $Fun(G^{\otimes n}, B) = \mathrm{Hom}_R(H^{\otimes n}, B)$.

Example 3.6. If we let G be the classical vertex group then we showed in section 2.2 that the collection of nonsingular functions, $\mathrm{Hom}_{\mathbb{C}}(G, B)$, corresponds to $B[\![x]\!]$. Since this can not be localised, we have $Fun(G, B) = \mathrm{Hom}_{\mathbb{C}}(G, B) \cong B[\![x]\!]$. The G-invariant maps are given by $Fun_G(G, B) = \mathrm{Hom}_G(G, B) \cong B$.

Example 3.7. Again letting G be the classical vertex group, we have also shown that the collection of nonsingular functions from $G^{\otimes 2}$ to B corresponds to $B[\![x_1, x_2]\!]$, which we consider as a module over $\mathbb{C}[\![x_1, x_2]\!]$. The ring of singular functions on G is isomorphic to $\mathbb{C}[\![z]\!][z^{-1}]$. We can only localise at $(1, 2)$ so consider the dual of the map $\widehat{f}_{12} : H^{\otimes 2} \to H$. For any $D^{(a)}, D^{(b)} \in H$, this gives $\widehat{f}_{12}(D^{(a)} \otimes D^{(b)}) = \binom{a+b}{a}(-1)^b D^{(a+b)}$, so the dual map $f_{12} : \mathbb{C}[\![z]\!] \to \mathbb{C}[\![x_1, x_2]\!]$ takes $z^k \in H^*$ to

$$\begin{aligned} f_{12}(z^k) &= \sum_{p+q=k} \binom{p+q}{p} (-1)^q x_1^p x_2^q \\ &= (x_1 - x_2)^k. \end{aligned}$$

This map can be extended from H^* to include negative values of k, in which case we will have infinitely many nonzero summands for negative values of q, and will still have $f_{12}(z^k) = (x_1 - x_2)^k$. Therefore our space of singular functions from $G^{\otimes 2}$ to B is given by

$$\begin{aligned} Fun(G^{\otimes 2}, B) &= B[\![x_1, x_2]\!] \otimes_{H^*} \mathbb{C}[\![z]\!][z^{-1}] \\ &\cong B[\![x_1, x_2]\!][\![(x_1 - x_2)]\!][(x_1 - x_2)^{-1}] \\ &= B[\![x_1, x_2]\!][(x_1 - x_2)^{-1}]. \end{aligned}$$

The G-invariant subcollection of maps in $Fun(G^{\otimes 2}, B)$ is the quotient of $B[\![x_1, x_2]\!][(x_1 - x_2)^{-1}]$ by the relation $D_B^{(1)} - \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}$, where $D_B^{(1)}$ is the action of $D^{(1)}$ on the module B. We thus have $Fun_G(G^{\otimes 2}, B) \cong B[\![x_1 - x_2]\!][(x_1 - x_2)^{-1}]$.

Example 3.8. Continuing the previous example for the case of $G^{\otimes 3}$, we know that the space of nonsingular functions from $G^{\otimes 3}$ to B corresponds to $B[\![x_1, x_2, x_3]\!]$ and we need to localise at the three ordered pairs (i,j) for $1 \le i < j \le 3$. The dual maps f_{12}, f_{13}, f_{23}, behave exactly as in the previous example, and so we have for our space of singular functions from $G^{\otimes 3}$ to B the collection

$$\begin{aligned} Fun(G^{\otimes 3}, B) &= B[\![x_1, x_2, x_3]\!] \otimes \mathbb{C}[\![(x_1 - x_2)]\!][(x_1 - x_2)^{-1}] \otimes \mathbb{C}[\![(x_1 - x_3)]\!][(x_1 - x_3)^{-1}] \\ &\qquad \otimes \mathbb{C}[\![(x_2 - x_3)]\!][(x_2 - x_3)^{-1}] \\ &\cong B[\![x_1, x_2, x_3]\!][(x_1 - x_2)^{-1}, (x_1 - x_3)^{-1}, (x_2 - x_3)^{-1}]. \end{aligned}$$

This can be extended in the obvious way to functions from $G^{\otimes n}$ to B giving:

$$Fun(G^{\otimes n}, B) \cong B[\![x_1, \ldots, x_n]\!][(x_i - x_j)^{-1}_{1 \le i < j \le n}]. \tag{3.3}$$

3.2 Singular Multilinear Maps

Now that we know how to use the elementary vertex structure of our vertex groups to add singularities to the picture, we can easily see how to generalise the extended representations of multilinear maps to define singular multilinear maps. But following the discussion given in section 1.1, we want to define these maps very delicately.

Consider first the collection of linear maps between G-modules A and B, $\mathrm{Hom}_R(A, B)$. We saw in section 2.4 that this collection is isomorphic to the collection of G-linear maps from A to $\mathrm{Hom}_R(G, B)$. Replacing the collection of nonsingular maps $\mathrm{Hom}_R(G, B)$ with the collection of singular maps $Fun(G, B)$ has no effect since they are equal, so we have:

Definition 3.9. *The singular multilinear maps from A to B are just the multilinear maps in the extended representation:*

$$Multi^K(A;B) = \mathrm{Hom}_G(A, \mathrm{Hom}(G,B)).$$

Repeating the same process for the collection of maps between G-modules $A_1 \otimes A_2$ and B, we have the following as our singular multi maps:

$$Multi^K(A_1, A_2; B) = \mathrm{Hom}_{G^{\otimes 2}}(A_1 \otimes A_2, Fun(G^{\otimes 2}, B)).$$

Lemma 3.10. *The G-invariant singular multilinear maps, $Multi_G^K(A_1, A_2; B)$ are the $G^{\otimes 2}$-invariant maps from $A_1 \otimes A_2$ to $Fun_G(G^{\otimes 2}, B)$.*

Proof. The $G^{\otimes 2}$-invariance of the singular multilinear maps allows the action of G on the domain, $A_1 \otimes A_2$, to carry over to an action on the domain of $Fun(G^{\otimes 2}, B)$ as described in section 2.4. Hence the action of G on $A_1 \otimes A_2$ passes through to an action on B exactly when $Fun(G^{\otimes 2}, B)$ is G-invariant. $\square$

Example 3.11. Take G to be the classical vertex group, and let f be a singular multilinear maps in $Multi^K(A;B) = \mathrm{Hom}_G(A, B[\![x]\!])$. Then for any $a \in A$ we can write $f(a) = \sum_{i \geq 0} b_i x^i$ for $b_i \in B$. If f is G-invariant, then $Tf(a) = \partial_x f(a)$, and so $Tb_i = (i+1)b_{i+1}$. Thus

$$f(a) = \sum_{i \geq 0} \frac{T^i}{i!} b_0 x^i = e^{Ty} b_0,$$

and so we have an isomorphism of G-modules,

$$Multi_G^K(A;B) \cong \mathrm{Hom}(A,B).$$

Example 3.12. With G taken to be the classical vertex group, the singular multilinear maps from $A_1 \otimes A_2$ to B are the $G^{\otimes 2}$-invariant maps from $A_1 \otimes A_2$ to $B[\![x,y]\!][(x-y)^{-1}]$. Given any such map, say f, the $G^{\otimes 2}$-invariance just says that for any $a \otimes b \in A \otimes B$,

$$\begin{aligned} f(Ta \otimes b) &= \partial_x f(a \otimes b) \\ f(a \otimes Tb) &= \partial_y f(a \otimes b). \end{aligned}$$

If we add the additional requirement that f be G-invariant, then we have

$$\begin{aligned} Tf(a \otimes b) &= f(Ta \otimes b) + f(a \otimes Tb) \\ &\quad \partial_x f(a \otimes b) + \partial_y f(a \otimes b), \end{aligned}$$

so G acts on $B[\![x,y]\!][(x-y)^{-1}]$ as $T_B = \partial_x + \partial_y$. Taking $V = A_1 = A_2 = B$, and setting

$$Y(\cdot, x)\cdot = f(\cdot \otimes \cdot)|_{y=0} : V \otimes V \longrightarrow V[\![x]\!][x^{-1}],$$

we have a vertex operator that satisfies the translation covariance axiom of equation (2.4).

Example 3.13. Having seen how to compute a vertex operator from a singular map, we now show how to compute a binary singular map from a vertex operator. Let $Y(\cdot, x)\cdot$ be a vertex operator on a complex vector space V. We claim that the map,

$$f(\cdot \otimes \cdot) = e^{Ty} Y(\cdot, x-y)\cdot,$$

gives the desired G-invariant binary singular map. (Here the operator $e^{Ty} = \sum_{i \geq 0} y^i D^{(i)}$ provides a linear map from V to $V[\![y]\!]$, and we are regarding $Y(\cdot, x-y) \in V[\![x-y]\!][(x-y)^{-1}]$ as an element of $V[\![x,y]\!][(x-y)^{-1}]$ under binomial expansion. We shall see in example 4.5 that this inclusion is

important for the general description that will follow.) For any $a \otimes b \in V \otimes V$, clearly we see that $f(a \otimes b)|_{y=0} = Y(a, x)b$, and

$$\begin{aligned} f(a \otimes b)|_{x=0} &= e^{Ty} Y(a, -y)b \\ &= Y(b, y)a \end{aligned}$$

where the second equality follows by quasisymmetry (see appendix C). Notice also that we could have just as easily reconstructed $f(a \otimes b)$ from $e^{Tx} Y(b, y - x)a$. Thus we see that a binary singular map incorporates the quasisymmetry of a vertex operator.

This example provides an interesting contrast with the case for holomorphic vertex algebras in section 2.4. There we saw that the entire theory was determined by a vertex operator $Y(\cdot, x)\cdot$ evaluated at $x = 0$. Because we have singularities here, we can not make such an evaluation, but we still can use the actions of G to reconstruct a singular map from the vertex operator.

The next obvious question is what happens when we compose two singular multilinear maps? We begin with an example using the classical vertex group:

Example 3.14. Following the discussion of composition in section 2.4, we shall assume that all maps are G-invariant, except possibly the bottom map. With an eye to emulating the behaviour of axiomatic vertex algebras, we shall also consider pointwise composition of these maps.

Taking the classical vertex group, G, the simplest nontrivial example of composition is the composition of $f \in \mathit{Multi}_G^K(A_1, A_2; B_1)$ with $h \in \mathit{Multi}^K(B_1, B_2; C)$. Writing these out explicitly, we have:

$$A_1 \otimes A_2 \xrightarrow{f} B_1[\![x_1, x_2]\!][(x_1 - x_2)^{-1}] \qquad\qquad B_1 \otimes B_2 \xrightarrow{h} C[\![z_1, z_2]\!][(z_1 - z_2)^{-1}]. \tag{3.4}$$

These compose to give a map

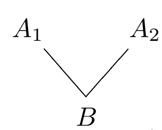
$$A_1 \otimes A_2 \otimes B_2 \xrightarrow{h \circ f} C[\![z_1, z_2]\!][(z_1 - z_2)^{-1}][\![x_1, x_2]\!][(x_1 - x_2)^{-1}].$$

By the G-invariance of f, we know that this $G^{\otimes 2}$-linear map satisfies $\partial_{z_1} = \partial_{x_1} + \partial_{x_2}$. Using these differential equations, the composite $h \circ f$ can be seen to factor through $C[\![X_1, X_3]\!][(X_1 - X_3)^{-1}][\![X_1 - X_2]\!][(X_1 - X_2)^{-1}]$ under either of the following changes of variables:

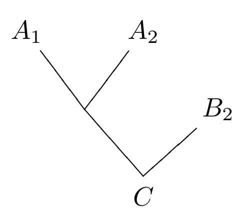
$$X_1 = x_1 + z_1 \qquad\qquad X_1 = x_2 + z_1 \tag{3.5}$$
$$X_2 = x_2 + z_1 \qquad\qquad X_2 = x_1 + z_1 \tag{3.6}$$
$$X_3 = z_2 \qquad\qquad X_3 = z_2. \tag{3.7}$$

We notice immediately that even though the composite factors through a collection which has only 3 variables, it does not factor through $\mathit{Fun}(G^{\otimes 3}, B) \cong B[\![x_1, x_2, x_3]\!][(x_1 - x_2)^{-1}, (x_1 - x_3)^{-1}, (x_2 - x_3)^{-1}]$. In fact it is simple to find examples of such composites which do not fit inside $\mathit{Fun}(G^{\otimes 3}, B)$. Thus our singular multilinear maps do not compose in the way we might have initially hoped.

Since we haven't yet defined singular multilinear maps for three input modules, $\mathit{Multi}^K(A_1, A_2, A_3; B)$, we could think of defining it to be a space large enough to contain the composite described in the previous example, and all other possible composites. But resulting spaces would end up simply being the spaces of formal distributions which we had initially set out to avoid. Instead, we shall define a collection of singular multilinear maps for every way of composing singular multilinear maps. In other words, if we denote the collection $\mathit{Multi}^K(A_1, A_2; B)$ by the labelled tree



then the composite would be an element of a space associated to the labelled tree



and so to every binary tree p, we would associate a collection of singular multilinear maps, $\mathit{Multi}_p^K(A_1, \dots, A_n; C)$. How then shall we define $\mathit{Multi}^K_{\setminus\!\!\!\vee}(A_1, A_2, A_3; C)$? Following the discussion of section 1.1 we shall take the following definition:

Definition 3.15. *The collection of multilinear singular maps from A_1, A_2, A_3 to C associated to the tree $\setminus\!\!\!\vee$ is given by the following equation:*

$$\mathit{Multi}^K_{\setminus\!\!\!\vee}(A_1, A_2, A_3; C) := \mathrm{Hom}\Big(A_1 \otimes A_2, K \otimes \mathrm{Hom}\big(G^{\otimes 2} \otimes A_3, K \otimes \mathrm{Hom}(G^{\otimes 2}, C)\big)\Big) / \sim,$$

where we are quotienting by four actions of G. These will be clear by regarding this collection of multi maps inside the larger collection:

$$\mathrm{Hom}_{G^{\otimes 3}}\Big(A_1 \otimes A_2 \otimes A_3, K \otimes \mathrm{Hom}_G\big(G^{\otimes 2}, K \otimes \mathrm{Hom}(G^{\otimes 2}, C)\big)\Big).$$

Then as with the binary singular maps described above, the action on each of the A_i carries over to an action on its respective occurrence of G, and the additional action of G acts on the one hand on the remaining occurrence of G, and on the other hand it acts diagonally on the outer $G^{\otimes 2}$ term.

Note. *Would it be better to use something like this:*

$$\begin{aligned} \mathit{Multi}^K_{\setminus\!\!\!\vee}(A_1, A_2, A_3; C) &:= \mathrm{Hom}_{G^{\otimes 3}}(A_1 \otimes A_2 \otimes R, K \otimes \mathrm{Hom}_{G^{\otimes 2}}(G^{\otimes 2} \otimes A_3, K \otimes \mathrm{Hom}(G^{\otimes 2}, C))) \\ &= \mathrm{Hom}_{G^{\otimes 2}}(A_1 \otimes A_2, \mathit{Fun}_{G^{\otimes 2}}(G^{\otimes 2} \otimes A_3, \mathit{Fun}(G^{\otimes 2}, C))). \end{aligned}$$

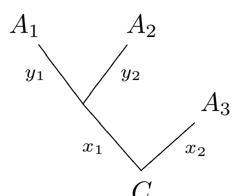
Example 3.16. Taking G to be the classical vertex group, this just says that the collection of singular multilinear maps, $\mathit{Multi}^K_{\setminus\!\!\!\vee}(A_1, A_2, A_3; C)$ is the subcollection of

$$\mathrm{Hom}\Big(A_1 \otimes A_2, \mathrm{Hom}\big(A_3, C[\![x_1, x_2]\!][(x_1 - x_2)^{-1}]\big)[\![y_1, y_2]\!][(y_1 - y_2)^{-1}]\Big),$$

satisfying the equations

$$\begin{aligned} T_{A_1} &= \partial_{y_1}, \\ T_{A_2} &= \partial_{y_2}, \\ T_{A_3} &= \partial_{x_2}, \text{ and} \\ \partial_{y_1} + \partial_{y_2} &= \partial_{x_1}, \end{aligned}$$

where T_{A_i} denotes the action of G on the module, A_i. We can represent this subcollection pictorially by the labelled tree

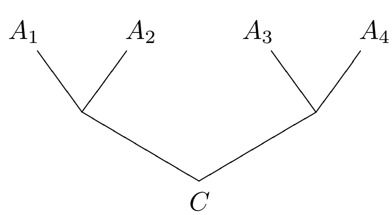


or consider it to be the appropriate subcollection of

$$\mathrm{Hom}\Big(A_1 \otimes A_2 \otimes A_3, C[\![x_1, x_2]\!][(x_1 - x_2)^{-1}][\![y_1, y_2]\!][(y_1 - y_2)^{-1}]\Big),$$

where the singularity $(y_1 - y_2)^{-1}$ is "independent" of A_3.

We can extend this definition naturally to include all binary trees that have exactly one splitting at each level. A difficulty arises when we consider two compositions at a single level. For example, the collection of singular maps associated to the binary labelled tree:



We could arrive at a singular multilinear map of type ${}^{\vee}{\vee}^{\vee}$ via composition in three ways: for G-modules B_1 and B_2, we could compose three binary singular maps:

$$Multi^K_{\vee}(A_1, A_2; B_1)\otimes Multi^K_{\vee}(A_3, A_4; B_2)\otimes Multi^K_{\vee}(B_1, B_2; C) \longrightarrow Multi^K_{{}^{\vee}{\vee}^{\vee}}(A_1, A_2, A_3, A_4; C), \tag{3.8}$$

or we could compose an appropriate binary singular map with a singular map of type ${}^{\vee}{\vee}$ or ${\vee}^{\vee}$ as:

$$\begin{aligned} Multi^K_{{}^{\vee}{\vee}}(A_1, A_2, B_2; C)\otimes Multi^K_{\vee}(A_3, A_3; B_2) &\longrightarrow Multi^K_{{}^{\vee}{\vee}^{\vee}}(A_1, A_2, A_3, A_4; C) && (3.9)\\ Multi^K_{\vee}(A_1, A_2; B_1)\otimes Multi^K_{{\vee}^{\vee}}(B_1, A_3, A_4; C) &\longrightarrow Multi^K_{{}^{\vee}{\vee}^{\vee}}(A_1, A_2, A_3, A_4; C). && (3.10)\end{aligned}$$

We would like to end up with a space which consists exactly of maps that could arise as composites. The important feature of this definition is the restriction of the dependence of each singularity to the relevant controlling modules. This suggests the following definition:

Definition 3.17. *The collection of singular multilinear maps of type ${}^{\vee}{\vee}^{\vee}$ from A_1, A_2, A_3, A_4 to C is defined to be the pullback of the singular multilinear maps,*

$$\mathrm{Hom}_{G,G^{\otimes 2}}\Big(A_1\otimes A_2, K\otimes\mathrm{Hom}_{G,G^{\otimes 2}}\big(A_3\otimes A_4, K\otimes\mathrm{Hom}_{G,G^{\otimes 2}}(G^{\otimes 4}, K\otimes\mathrm{Hom}(G^{\otimes 2}, C))\big)\Big) \tag{3.11}$$

$$\mathrm{Hom}_{G,G^{\otimes 2}}\Big(A_3\otimes A_4, K\otimes\mathrm{Hom}_{G,G^{\otimes 2}}\big(A_1\otimes A_2, K\otimes\mathrm{Hom}_{G,G^{\otimes 2}}(G^{\otimes 4}, K\otimes\mathrm{Hom}(G^{\otimes 2}, C))\big)\Big) \tag{3.12}$$

over

$$\mathrm{Hom}_{G^{\otimes 4}}\Big(A_1\otimes A_2\otimes A_3\otimes A_4, K\otimes K\otimes\mathrm{Hom}_{G^{\otimes 2}}\big(G^{\otimes 4}, K\otimes\mathrm{Hom}(G^{\otimes 2}, C)\big)\Big) \tag{3.13}$$

where the appropriate G-invariance is taken into account.

Note. *Because invariance just appears as an equaliser, it commutes with pullbacks.*

Theorem 3.18. *The composites of singular maps in equations* (3.8)*,* (3.9)*, and* (3.10) *compose as desired.*

Before we prove this theorem, we need the following lemma.

Lemma 3.19. *In any symmetric monoidal category, $\mathcal{C}$, the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Hom}(A_1, B_1\otimes C_1)\otimes\mathrm{Hom}(A_2, B_2\otimes C_2) \arrow[r] \arrow[d] & \mathrm{Hom}(A_1, C_1\otimes\mathrm{Hom}(A_2, B_1\otimes B_2\otimes C_2)) \arrow[d] \\
\mathrm{Hom}(A_2, C_2\otimes\mathrm{Hom}(A_1, B_1\otimes C_1\otimes B_2)) \arrow[r] & \mathrm{Hom}(A_1\otimes A_2, B_1\otimes C_1\otimes B_2\otimes C_2)
\end{tikzcd}$$

Proof. The proof follows immediately from the fact that the evaluation of

$$\mathrm{Hom}(A_1, B_1\otimes C_1)\otimes\mathrm{Hom}(A_2, B_2\otimes C_2)$$

on $A_1\otimes A_2$ gives the same result when carried out by either first evaluating A_1, or by first evaluating A_2 or by evaluating both together. $\square$

Proof of theorem 3.18. We shall only prove that the composition works for the classical vertex group. For the general theory, see [15]. Since composition is pointwise, the composite (3.8) factors through both (3.9) and (3.10), and so we can focus on those compositions. In order to prove that the composites map into the pullback, we shall first prove that they map into each of the pullback objects (the singular maps given in (3.11) and (3.12)).

Taking the composite in equation (3.9), we see that it can be considered a multi map as in equation (3.11) through the following natural map:

$$
\begin{aligned}
&\mathit{Multi}^K_{\vee\!\!\!\vee}(A_1, A_2, B_2; C) \otimes \mathit{Multi}^K_{\vee}(A_3, A_3; B_2) = \\
&\mathrm{Hom}\Big(A_1 \otimes A_2, K \otimes \mathrm{Hom}\big(G^{\otimes 2} \otimes B_2, K \otimes \mathrm{Hom}(G^{\otimes 2}, C)\big)\Big) \otimes \mathrm{Hom}\Big(A_3 \otimes A_4, K \otimes \mathrm{Hom}(G^{\otimes 2}, B_2)\Big) \\
&\longrightarrow \mathrm{Hom}\bigg(A_1 \otimes A_2, K \otimes \mathrm{Hom}\big(G^{\otimes 2} \otimes B_2, K \otimes \mathrm{Hom}(G^{\otimes 2}, C)\big) \otimes \mathrm{Hom}\Big(A_3 \otimes A_4, K \otimes \mathrm{Hom}(G^{\otimes 2}, B_2)\Big)\bigg) \\
&\longrightarrow \mathrm{Hom}\bigg(A_1 \otimes A_2, K \otimes \mathrm{Hom}\Big(A_3 \otimes A_4, K \otimes \mathrm{Hom}(G^{\otimes 2}, \mathrm{Hom}\big(G^{\otimes 2}, K \otimes \mathrm{Hom}(G^{\otimes 2}, C)\big))\Big)\bigg) \\
&\cong \mathrm{Hom}\bigg(A_1 \otimes A_2, K \otimes \mathrm{Hom}\Big(A_3 \otimes A_4, K \otimes \mathrm{Hom}(G^{\otimes 4}, K \otimes \mathrm{Hom}(G^{\otimes 2}, C))\Big)\bigg).
\end{aligned}
$$

Similarly we can can see that it gives a multi map as in equation (3.12) through evaluating $A_3 \otimes A_4$ first. From the lemma, we know that these two ways of evaluating are equal, so the composite must factor through the pullback. □

In the next section we give a general definition for singular multilinear maps parameterised by any binary tree. But before we do so, we finish this section with a description of $\mathit{Multi}^K_{\vee\vee}(A_1, A_2, A_3, A_4; C)$ for the classical vertex group.

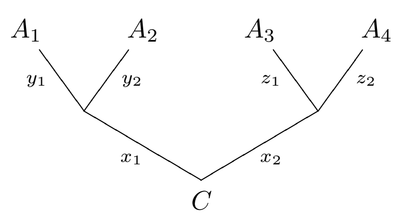
Example 3.20. Letting G be the classical vertex group, the collection of singular multilinear maps of type $\vee\vee$ from A_1, A_2, A_3, A_4 to C is the submodule of

$$\mathrm{Hom}\Big(A_1 \otimes A_2 \otimes A_3 \otimes A_4, C[\![x_1, x_2]\!][(x_1 - x_2)^{-1}][\![y_1, y_2, z_1, z_2]\!][(y_1 - y_2)^{-1}, (z_1 - z_2)^{-1}]\Big),$$

where the singularity $(y_1 - y_2)^{-1}$ is independent of $A_3 \otimes A_4$, the singularity $(z_1 - z_2)^{-1}$ is independent of $A_1 \otimes A_2$, and the maps satisfy the following equations:

$$
\begin{aligned}
T_{A_1} &= \partial_{y_1} & \qquad T_{A_2} &= \partial_{y_2} \\
T_{A_3} &= \partial_{z_1} & \qquad T_{A_4} &= \partial_{z_2} \\
\partial_{y_1} + \partial_{y_2} &= \partial_{x_1} & \qquad \partial_{z_1} + \partial_{z_2} &= \partial_{x_2}
\end{aligned}
$$

We represent this collection pictorially by the following labelled tree:



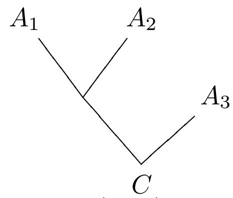
The G-invariant such maps are the further subcollection satisfying $T_C = \partial_{x_1} + \partial_{x_2}$.

3.3 Multi Maps Parameterised by Binary Trees

In this section we are concerned with giving the definition of a multilinear singular map associated to an arbritrary binary tree. Given any binary tree p, we may consider its collection of internal vertices. For our purposes, we shall assume that these include the root, but they do not include the leaves. Considering them as a set, this set inherits a partial order from the tree, where the root is the least element. We know that any partial order can be extended to at least one total ordering, possibly many.

We have already been working with trees whose leaves and root are labelled by G-modules. It will be useful for the explanations to follow to assume that every internal node is also labelled. For any internal node, q, connected to n **incoming** nodes (i.e., non-empty nodes whose height is equal to the height of q plus one and connected to q by a single edge), we shall label q by $G^{\otimes n}$. We can also associate to q the

tensor product of the labels of the incoming nodes, and denote it X_q. Thus the following labelled tree has two internal nodes,



and we have $X_{\text{root}} = G^{\otimes 2} \otimes A_3$, and $X_{\text{internal}} = A_1 \otimes A_2$.

We say that our tree is **augmented** when we have added an additional vertex and edge to the root of the tree, such that the new vertex inherits the former root label. The former root is labelled $G^{\otimes 2}$ as expected. We denote the new vertex $\perp$, and we automatically have $X_\perp = G^{\otimes 2}$.

Definition 3.21. *Let p be a binary n-leafed tree, and let t denote a total ordering, $\perp < root < p_1 < \cdots < p_l$, of the internal vertices of augmented p, compatible with the the partial ordering inherited from the tree structure of p. We define an operator on $G^{\otimes 2}$-modules:*

$$\mathrm{Sing}_{p_i} = \mathrm{Hom}_{G, G^{\otimes 2}}(X_{p_i}, K \otimes \cdot).$$

Iterating this operator we have

$$\mathrm{Ord}_t(A_1, \ldots, A_n, C) = \mathrm{Sing}_{p_l} \cdots \mathrm{Sing}_{p_1} \mathrm{Sing}_{root} \mathrm{Hom}(X_\perp, C).$$

Then $Multi_p^K(A_1, \ldots, A_n, C)$ is defined to be the wide pullback of each Ord_t for all possible total orderings, t, of of the internal vertices of augmented p, over

$$\mathrm{Hom}_{G^{\otimes n}}(A_1 \otimes \cdots \otimes A_n, Fun_p(G^{\otimes n}, C))$$

where $Fun_p(G^{\otimes n}, C))$ is the collection of singular functions as defined in [3]. See appendix D for more details.

*We have G-**invariant multilinear singular maps**, $Multi_{G,p}^K$ exactly when $\mathrm{Hom}(X_\perp, C)$ is G-invariant.*

Example 3.22. If p is a tree with only one binary tree at each level, then there is only one total ordering, t, of internal vertices of the tree, and so

$$Multi_p^K(A_1, \ldots, A_n, C) = \mathrm{Ord}_t(A_1, \ldots, A_n, C).$$

Example 3.23. When $p =$ , there are exactly two total orderings of internal vertices of this tree, and the corresponding $\mathrm{Ord}_t(A_1, \ldots, A_4, C)$ functions are given by

$$\mathrm{Ord}_{t_1} = \mathrm{Hom}\Big(A_1 \otimes A_2, K \otimes \mathrm{Hom}\big(A_3 \otimes A_4, K \otimes \mathrm{Hom}(G^{\otimes 4}, K \otimes \mathrm{Hom}(G^{\otimes 2}, C))\big)\Big) \tag{3.14}$$

$$\mathrm{Ord}_{t_2} = \mathrm{Hom}\Big(A_3 \otimes A_4, K \otimes \mathrm{Hom}\big(A_1 \otimes A_2, K \otimes \mathrm{Hom}(G^{\otimes 4}, K \otimes \mathrm{Hom}(G^{\otimes 2}, C))\big)\Big), \tag{3.15}$$

where the Hom functions are both G-invariant and $G^{\otimes 2}$-invariant. Since we are pulling back over

$$\mathrm{Hom}_{G^{\otimes 4}}(A_1 \otimes \cdots \otimes A_4, Fun_p(G^{\otimes 4}, C)) = \\ \mathrm{Hom}_{G^{\otimes 4}}(A_1 \otimes \cdots \otimes A_4, K \otimes K \otimes \mathrm{Hom}_{G^{\otimes 2}}(G^{\otimes 4}, K \otimes \mathrm{Hom}(G^{\otimes 2}, C))), \tag{3.16}$$

this definition reduces to definition 3.17.

4 Vacuum and Locality - Extending from Binary Trees

Now that we have defined spaces of (generalised) vertex operators and their composites, we shall return our attention to the remaining axioms for a vertex algebra. In particular, we will be interested in incorporating the vacuum axioms of equations (2.1), (2.2), and (2.3) into our description, and making sense of the locality axiom of equation (2.5). But first, we return to the case of singular maps between G-modules A and B, and parameterise them by non-branching trees.

4.1 Non-branching Trees

When we first considered singular multilinear maps in section 3.2, we defined $\mathit{Multi}^K(A;B)$ to be $\mathrm{Hom}_G(A,\mathrm{Hom}(G,B))$ (see definition 3.9). Now that we have singular multilinear maps parameterised by trees, we shall think of this collection of maps as being associated to the flat tree with one leaf, $|$. (For a review of definitions related to trees, see appendix A Thus we have

$$\mathit{Multi}^K_{|}(A;B) = \mathit{Multi}^K(A;B) = \mathrm{Hom}_G(A,\mathrm{Hom}(G,B)).$$

In fact, we can extend the construction of Multi^K_p, described in definition 3.21 for binary trees, to trees with non-branching subtrees. We do this by simply extending the definition of Sing_q to those internal nodes q with only one incoming edge.

Definition 4.1. *If the tree p in definition 3.21 is allowed to also have non-branching subtrees, then $\mathit{Multi}^K_p(A_1,\dots,A_n;C)$ is defined exactly as in that definition except that when an internal vertex, p_i has only one incoming edge, we define an operator to act on G-modules,*

$$\mathrm{Sing}_{p_i} = \mathrm{Hom}_G(X_{p_i},\cdot),$$

where X_{p_i} is the label of the incoming node as in section 3.3.

Lemma 4.2. *The definition of $\mathit{Multi}^K_{|}(A;B)$ coincides with definition 4.1 applied to the tree $|$.*

Proof. Repeating the construction of definition 4.1 for the tree $|$, we have

$$\mathit{Multi}^K_{|}(A;B) = \mathrm{Hom}_G(A,\mathrm{Hom}(G,B)) \cong \mathrm{Hom}(A,B)$$

as defined above. We also see that the G-invariant maps are the collection,

$$\mathit{Multi}^K_{G,|}(A;B) = \mathrm{Hom}_G(A,\mathrm{Hom}_G(G,B)) \cong \mathrm{Hom}_G(A,B).$$

$\square$

Example 4.3. If p is the non-branching tree with n internal nodes (including the root) then

$$\mathit{Multi}^K_p(A;B) = \mathrm{Hom}_G(A,\left(\mathrm{Sing}_{\,|}\right)^{(n-1)}\mathrm{Hom}(G,B)).$$

Taking advantage of the G-invariance of $\mathrm{Sing}_{\,|}$, we see that this collection is isomorphic to $\mathrm{Hom}_G(A,\mathrm{Hom}(G,B))$ in the usual way.

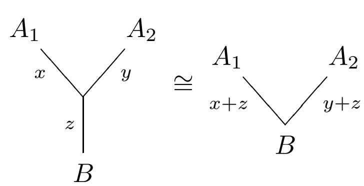
Example 4.4. If we take G to be the classical vertex group, then $\mathit{Multi}^K_{|}(A;B) \cong \mathrm{Hom}_G(A,B[\![x]\!])$. (c.f., example 3.11.)

For the non-branching tree with n internal nodes, p, we have $\mathit{Multi}^K_p(A;B) \cong \mathrm{Hom}(A,B[\![x_1,\dots,x_n]\!])$, where the G-invariance of $\mathrm{Sing}_{\,|}$ means that any such map is an element of the subcollection, $\mathrm{Hom}(A,B[\![x_1+\cdots+x_n]\!])$.

Example 4.5. Consider the singular maps associated to the tree, Y. From the definition we have

$$\begin{aligned}\mathit{Multi}^K_{\mathsf{Y}}(A_1,A_2;B) &= \mathrm{Hom}_{G,G^{\otimes 2}}\Big(A_1\otimes A_2, K\otimes\mathrm{Hom}_G\big(G^{\otimes 2},\mathrm{Hom}(G,B)\big)\Big)\\ &\cong \mathrm{Hom}_{G,G^{\otimes 2}}\Big(A_1\otimes A_2, K\otimes\mathrm{Hom}\big(G^{\otimes 2},B\big)\Big).\end{aligned}$$

When G is the classical vertex group this says that there is a bijection between the collection of multilinear singular maps associated to the following trees:



(4.1)

This isomorphism follows immediately from the G-invariance at the internal node, where it provides the relation $\partial_x + \partial_y = \partial_z$. The map between these spaces is just given by binomial expansion as in example 3.13, and the singularity remains unaffected. When these maps are fully G-invariant, the single edge attached to the bottom of the tree corresponds to operating on each singular map with

$$e^{Tz} = \sum_{i \geq 0} D^{(i)} z^i : B \longrightarrow B[\![z]\!].$$

We finish this section by noting in passing that composing any tree with the tree consisting of a single node, $\bullet$, leaves the tree unchanged. So we would like to define $\mathit{Multi}_{\bullet}^{K}$ so that it composes with a singular map of type p (for some tree, p) to give a singular map of type p. This suggests the following definition:

Definition 4.6. *For any G-module, A, the singular multilinear maps associated to the tree $\bullet$ are just the endomorphisms of A.*

$$\mathit{Multi}_{\bullet}^{K}(A; A) = \operatorname{Hom}(A, A).$$

Lemma 4.7. *The definition of $\mathit{Multi}_{\bullet}^{K}(A; A)$ coincides with definition 4.1 applied to the tree $\bullet$.*

Proof. Definition 4.1 applied to $\bullet$ gives $\mathit{Multi}_{\bullet}^{K}(A; A) = \operatorname{Hom}(A, A))$ as desired. □

4.2 Vacuum - Trees with no Leaves

Now that we have a description of unary singular maps, it is natural to consider the nullary type multi maps. By way of motivation, recall that the vacuum was defined to be a distinguished vector denoted $|0\rangle \in V$ such that for any $a \in V$, the vacuum satisfies:

$$\begin{aligned} T|0\rangle &= 0 \\ Y(|0\rangle, x)a &= a \\ Y(a, x)|0\rangle|_{x=0} &= a. \end{aligned}$$

We saw in section 2.4 that for vertex algebras without singularities, the vacuum acted as a unit object. Later we shall show that even with singularities, the vacuum will continue to act as a unit. But in this section we are concerned primarily with showing how the collection of all possible vacuum vectors arises naturally when considering singular maps parameterised by trees.

They arise naturally as the multilinear singular map associated to the empty tree (which we denote $\circ$). Applying definition 4.1, to the case of the empty tree, we construct the collection of multilinear singular maps by first augmenting the empty tree, giving $\overset{\circ}{\underset{\bullet}{|}}$. As with the tree $\bullet$, the only internal node is $\perp$. Thus we have $X_{\perp} = R$ since it is not connected to any other leaf or internal node (the empty node is not counted as a leaf), and so we have:

Definition 4.8. *The multilinear singular maps parameterised by the empty tree are given by:*

$$\mathit{Multi}_{\circ}^{K}(R; V) = \operatorname{Hom}(R, V) \cong V.$$

The G-invariant elements of this collection are the vectors in V with trivial G-action, so we see that a G-invariant singular multilinear map of type $\circ$ is just a vector $v \in V$ satisfying the first vacuum axiom.

What happens when the vacuum vector is used as an input for a binary singular multi map? We begin with an example for the classical vertex group.

Example 4.9. Let G be the classical vertex group and f be a binary singular map in

$$\mathit{Multi}_{\vee}^{K}(A_1, A_2; B) \cong \operatorname{Hom}_{G^{\otimes 2}}(A_1 \otimes A_2, B[\![x, y]\!][(x - y)^{-1}]).$$

We know that for a vector $v \in A_1$ arising as above, we have $Tv = 0$, so given any $a \in A_2$ we have,

$$\partial_x f(v \otimes a) = f(Tv \otimes a) = 0.$$

So $f(v\otimes a) \in B[\![y]\!]$. We see further what if we assume f to be G-invariant, we have that $Tf(v\otimes a) = \partial_y f(v\otimes a)$, and so writing $f(v\otimes a) = \sum_{i\geq 0} b_i y^i$, we see that $Tb_i = (i+1)b_{i+1}$, and so we have

$$f(v\otimes a) = \sum_{i\geq 0} \frac{T^i}{i!} b_0 y^i = e^{Ty} b_0. \tag{4.2}$$

But we saw in the previous section that $\mathit{Multi}^K_{\,!}(A_2; B) \cong \mathrm{Hom}_G(A_2, B[\![y]\!])$, and the G-invariant maps in this collection were of the form of equation (4.2). Thus we see that we have a composition map

$$\mathit{Multi}^K_{\vee}(A_1, A_2; B) \otimes \mathit{Multi}^K_{G,\circ}(R; A_1) \longrightarrow \mathit{Multi}^K_{\,!}(A_2; B), \tag{4.3}$$

and G-invariant binary singular maps compose to give G-invariant maps $\mathit{Multi}^K_{G,!}(A_2; B)$. Notice that if $A_1 = A_2 = B = V$ for some complex vector space V, then equation (4.2) gives the remaining vacuum axioms when $b_0 = a$. We shall see later that this will hold for a suitable algebra.

This is a very satisfactory notion for composition because we have taken a binary tree and composed it with the empty node to give a tree with only one leaf. In fact, this holds equally well for an arbritrary vertex group, with equation (4.3) holding for any G-modules A_1, A_2 and B. For the general theory see [15].

4.3 Locality

We have attempted to be quite clear throughout this paper as to the ramifications of our description for the classical vertex group. We would now like to apply the locality axiom to the classical vertex group story so far, and interpret it in the abstract presentation. Recall that the locality axiom says that for any $a, b \in V$, there exists some $N \gg 0$ such that the following holds:

$$(x-y)^N [Y(a,x), Y(b,y)] = 0.$$

In other words, the following two maps are equal:

$$\begin{array}{rcl} (x-y)^N Y(a,x)Y(b,y)\cdot : V & \longrightarrow & V[\![x]\!][x^{-1}][\![y]\!][y^{-1}] \\ (x-y)^N Y(b,y)Y(a,x)\cdot : V & \longrightarrow & V[\![y]\!][y^{-1}][\![x]\!][x^{-1}]. \end{array}$$

In light of our previous discussion showing how $Y(a,x)Y(b,y)\cdot$ and $Y(b,y)Y(a,x)\cdot$ are actually elements of different spaces of singular maps, this presents us with the question of how to interpret such an equality of maps.

The usual way to interpret them is by considering them inside the space of formal power series $V[\![x, x^{-1}, y, y^{-1}]\!]$. But since both of the spaces of power series in equations (4.4) and (4.4) are properly contained in the larger space of formal power series, they can only be equal if they map to the intersection $V[\![y]\!][y^{-1}][\![x]\!][x^{-1}] \cap V[\![x]\!][x^{-1}][\![y]\!][y^{-1}]$.

Lemma 4.10. $V[\![y]\!][y^{-1}][\![x]\!][x^{-1}] \cap V[\![x]\!][x^{-1}][\![y]\!][y^{-1}] = V[\![x,y]\!][x^{-1}, y^{-1}]$.

Proof. It is clear that the right hand side is contained in the intersection, so we need only to prove that an arbitrary element of the intersection is contained in the power series on the right. But the only difference between the power series $V[\![x]\!][x^{-1}][\![y]\!][y^{-1}]$, and $V[\![x,y]\!][x^{-1}, y^{-1}]$, is that in the former, polynomials in the variable x^{-1} can exist as coefficients of the power series in the variable y. But this can not occur in $V[\![y]\!][y^{-1}][\![x]\!][x^{-1}]$, so the intersection is as given. □

So we see that for some $N \gg 0$, $(x-y)^N Y(a,x)Y(b,y)\cdot$ and $(x-y)^N Y(b,y)Y(a,x)\cdot$ are equal as maps from V to $V[\![x,y]\!][x^{-1}, y^{-1}]$. Fixing N, we denote this map $g(a\otimes b\otimes\cdot)$.

Considering g as a map to the larger space, $V[\![x,y]\!][x^{-1}, y^{-1}, (x-y)^{-1}]$, of power series with the inverse of $(x-y)$ adjoined. Then the map, $(x-y)^{-N} g(a\otimes b\otimes\cdot)$, is well defined on V. Notice first that this map is not necessarily equal to $Y(a,x)Y(b,y)\cdot$ or $Y(b,y)Y(a,x)\cdot$, because $(x-y)^{-N}$ does not exist

in the codomain of either of these maps. But, if we expand $(x-y)^{-N}$ as a power series in either the variable x or y, we have maps

$$\begin{tikzcd}[column sep=small] & V[\![x,y]\!][x^{-1},y^{-1},(x-y)^{-1}] \arrow[dl, "i_{x,y}"'] \arrow[dr, "i_{y,x}"] & \\ V[\![x]\!][x^{-1}][\![y]\!][y^{-1}] & & V[\![y]\!][y^{-1}][\![x]\!][x^{-1}] \end{tikzcd} \tag{4.4}$$

which are identities except on $(x-y)^{-1}$, where they are given by

$$\begin{aligned} i_{x,y}\big((x-y)^{-j-1}\big) &= \sum_{n\in\mathbb{N}} \binom{n+j}{j} x^{-n-j-1}y^n \\ i_{y,x}\big((x-y)^{-j-1}\big) &= (-1)^{j+1}\sum_{n\in\mathbb{N}} \binom{n+j}{j} x^{n}y^{-n-j-1}, \end{aligned}$$

then we see that under these maps, the singular map $(x-y)^{-N}g(a\otimes b\otimes\cdot)$ canonically maps down to the two composites of vertex operators:

$$\begin{aligned} i_{x,y}\big((x-y)^{-N}g(a\otimes b\otimes\cdot)\big) &= Y(a,x)Y(b,y)\,\cdot \\ i_{y,x}\big((x-y)^{-N}g(a\otimes b\otimes\cdot)\big) &= Y(b,y)Y(a,x)\,\cdot\,. \end{aligned}$$

So we have constructed an element of the following collection of singular maps (with appropriate G-invariance taken):

$$\mathrm{Hom}\Big(V\otimes V, \mathrm{Hom}\Big(V, V[\![x,y]\!][x^{-1},y^{-1}]\Big)[(x-y)^{-1}]\Big),$$

where we have taken special care to emphasise that the singularity at $(x-y)$ depends only upon the outer two copies of V. We shall see shortly that this dependence of singularities on inputs is a crucial feature of the more general theory. But first we examine the effect of the vacuum on these composites.

Example 4.11. Consider the action of $(x-y)^{-N}g(a\otimes b\otimes\cdot)$ on the vacuum, $|0\rangle$. From its definition we have

$$\begin{aligned} (x-y)^{-N}g(a\otimes b\otimes|0\rangle) &= (x-y)^{-N}(x-y)^{N}Y(a,x)Y(b,y)|0\rangle \\ &= (x-y)^{-N}(x-y)^{N}Y(b,y)Y(a,x)|0\rangle, \end{aligned}$$

and we know from the vacuum axioms that $Y(a,x)|0\rangle$ and $Y(b,y)|0\rangle$ have no singularities, therefore $(x-y)^{-N}g(a\otimes b\otimes|0\rangle)$ is an element of $V[\![x,y]\!][(x-y)^{-1}]$. From our previous discussion we know that it maps down to the composites

$$\begin{aligned} i_{x,y}\big((x-y)^{-N}g(a\otimes b\otimes|0\rangle)\big) &= Y(a,x)e^{Ty}b \\ i_{y,x}\big((x-y)^{-N}g(a\otimes b\otimes|0\rangle)\big) &= Y(b,y)e^{Tx}a, \end{aligned}$$

which are elements of $V[\![x]\!][x^{-1}][\![y]\!]$ and $V[\![y]\!][y^{-1}][\![x]\!]$ respectively. Recall that in example 3.13 we showed how to construct a singular map from a vertex operator.

Lemma 4.12. *The binary singular power series given by regarding $e^{Ty}Y(a,x-y)b$ as an element of $V[\![x,y]\!][(x-y)^{-1}]$ is equal to $(x-y)^{-N}g(a\otimes b\otimes|0\rangle)$.*

Proof. Each of these is an element of $V[\![x,y]\!][(x-y)^{-1}]$ which agree when $x=0$ (and when $y=0$). Because each satisfies $\partial_x+\partial_y=\partial_V$ they are completely determined by their values at $x=0$ (or $y=0$), and so are equal. $\square$

Just as we looked at the action of $(x-y)^{-N}g(a\otimes b\otimes\cdot)$ on the vacuum, we consider its action on an arbitrary $c\in V$. From the vacuum axioms, we know that $c=Y(c,z)|0\rangle|_{z=0}$, so in fact we shall consider

$$(x-y)^{-N}g\Big(a\otimes b\otimes Y(c,z)|0\rangle\Big)\Big|_{z=0} = (x-y)^{-N}(x-y)^{N}Y(a,x)Y(b,y)Y(c,z)|0\rangle|_{z=0}.$$

Using the locality axiom, we know that for some $M \gg 0$ the following equality holds for any $a \in V$:

$$(y-z)^M Y(a,x)Y(b,y)Y(c,z)|0\rangle = (y-z)^M Y(a,x)Y(c,z)Y(b,y)|0\rangle.$$

Setting $z = 0$, we see that as an element of $V[\![x,y]\!][(x-y)^{-1}, x^{-1}, y^{-1}]$, the singularity y^{-1} of $(x-y)^{-N} g(a \otimes b \otimes c)$ depends only on b and c. Similarly, the singularity x^{-1} depends only on a and c.

Recalling how when we defined the binary singular maps, we defined maps $f(a \otimes b)$ such that setting $y = 0$ gave $Y(a,x)b$ and setting $y = 0$ gave $Y(b,y)a$, we see that a more natural definition for g would be as a map to $V[\![x,y,z]\!][(x-y)^{-1}, (y-z)^{-1}, (x-z)^{-1}]$. For any $a, b, c \in V$ we may define it (with $N \gg 0$ as before):

$$(x-y)^{-N}(x-y)^N \Big(e^{Tz} Y(a, x-z) Y(b, y-z) c\Big).$$

Thus we have a ternary function which, according to the dependence of singularities on inputs discussed above, can we regarded as an element of:

$$\mathrm{Hom}\Big(V \otimes V, \mathrm{Hom}\Big(V, V[\![x,y,z]\!][(x-z)^{-1}, (y-z)^{-1}]\Big)[(x-y)^{-1}]\Big), \tag{4.5}$$

$$\mathrm{Hom}\Big(V \otimes V, \mathrm{Hom}\Big(V, V[\![x,y,z]\!][(x-z)^{-1}, (x-y)^{-1}]\Big)[(y-z)^{-1}]\Big), \tag{4.6}$$

$$\mathrm{Hom}\Big(V \otimes V, \mathrm{Hom}\Big(V, V[\![x,y,z]\!][(y-z)^{-1}, (x-y)^{-1}]\Big)[(x-z)^{-1}]\Big). \tag{4.7}$$

All three of these collections of singular maps are properly contained in the larger collection,

$$\mathrm{Hom}\Big(V \otimes V \otimes V, V[\![x,y,z]\!][(x-z)^{-1}, (y-z)^{-1}, (x-y)^{-1}]\Big), \tag{4.8}$$

where the dependence of each singularity on input is ignored. So we are led to define a new collection of multi maps which we associate to the flat three leafed tree, , as the pullback of the spaces in equations (4.5), (4.6), and (4.7) over the space in equation (4.8). We shall denote it $\mathit{Multi}^K(V,V,V;V)$. In fact, this collection formalises the idea of the **operator product expansion** (see [11, section 4.6]).

Note. *We have tacitly been using the isomorphism of trees of equation 4.1.*

4.4 Flat Trees and Singular Maps

In the previous section we used our knowledge of axiomatic vertex algebras to define a collection of multi maps associated to a three leaved flat tree. Following that example, we now provide a more general description of singular maps for an arbitrary vertex group associated to any n-leafed flat tree. This definition arises naturally from the previous discussion, and incorporates the definitions provided for trees with $0, 1$ and 2 leaves.

Definition 4.13. *If G is a vertex group and $A_1, \dots, A_n, B$ are G-modules, then the collection of singular maps associated to the flat tree with n leaves, , is denoted $\mathit{Multi}^K(A_1, \dots, A_n; B)$, and is defined to be the pullback of the following singular maps for $\sigma \in A_n$ (the alternating group):*

$$\mathrm{Sing}_\sigma(A_1, \dots, A_n; \mathrm{Hom}(G^{\otimes n}, B)) =$$
$$\mathrm{Hom}\Big(A_{\sigma(1)} \otimes A_{\sigma(2)}, K \otimes \mathrm{Hom}\Big(A_{\sigma(3)}, K^{\otimes 2} \otimes \big(\cdots K^{\otimes n-1} \otimes \mathrm{Hom}(G^{\otimes n}, B) \cdots\big)\Big)\Big), \tag{4.9}$$

over the collection,

$$\mathrm{Hom}_{G^{\otimes n}}\Big(A_1 \otimes \cdots \otimes A_n, \mathit{Fun}(G^{\otimes n}, B)\Big) = \mathrm{Hom}_{G^{\otimes n}}\Big(A_1 \otimes \cdots \otimes A_n, K^{\otimes \binom{n}{2}} \otimes \mathrm{Hom}(G^{\otimes n}, B)\Big), \tag{4.10}$$

*where equation (4.9) is taken to be invariant under the action of $G^{\otimes n}$ inferred from its action on equation (4.10). The singularities of equation (4.9) are the singularities are tensored over $f_{i,j}$ as in definition 3.3, of $\mathit{Fun}(G^{\otimes n}, B)$. We have **G-invariant multilinear singular maps** exactly when $\mathrm{Hom}(G^{\otimes n}, B)$ is G-invariant.*

Example 4.14. When $n = 0$, this definition says that $\mathit{Multi}^K_{\circ}(R; B) = \mathrm{Hom}(R, B)$ as expected, and when $n = 1$, we have that $\mathit{Multi}^K_{|}(A; B) = \mathrm{Hom}_G(A, \mathrm{Hom}(G, B))$ as expected.

Example 4.15. When $n = 2$, we see that $\mathit{Multi}^K_{\vee}(A_1, A_2; B) = \mathrm{Hom}_{G^{\otimes 2}}(A_1 \otimes A_2, K \otimes \mathrm{Hom}(G^{\otimes 2}, B))$ as before.

Example 4.16. If G is a vertex group and A_1, A_2, A_3, B are G-modules, then the collection of singular maps, $\mathit{Multi}^K_{\vee\!\vee}(A_1, A_2, A_3; B)$, is the pullback of the following three collections of singular maps:

$$\mathrm{Hom}\Big(A_1 \otimes A_2, K \otimes \mathrm{Hom}\Big(A_3, K^{\otimes 2} \otimes \mathrm{Hom}(G^{\otimes 3}, B)\Big)\Big), \tag{4.11}$$

$$\mathrm{Hom}\Big(A_2 \otimes A_3, K \otimes \mathrm{Hom}\Big(A_1, K^{\otimes 2} \otimes \mathrm{Hom}(G^{\otimes 3}, B)\Big)\Big), \tag{4.12}$$

$$\mathrm{Hom}\Big(A_3 \otimes A_1, K \otimes \mathrm{Hom}\Big(A_2, K^{\otimes 2} \otimes \mathrm{Hom}(G^{\otimes 3}, B)\Big)\Big), \tag{4.13}$$

over the collection,

$$\mathrm{Hom}_{G^{\otimes 3}}\Big(A_1 \otimes A_2 \otimes A_3, \mathit{Fun}(G^{\otimes 3}, B)\Big) = \mathrm{Hom}_{G^{\otimes 3}}\Big(A_1 \otimes A_2 \otimes A_3, K^{\otimes 3} \otimes \mathrm{Hom}(G^{\otimes 3}, B)\Big), \tag{4.14}$$

with appropriate G-invariance taken. We see immediately that the definition of $\mathit{Multi}^K_{\vee\!\vee}(V, V, V; V)$ for the classical vertex group follows immediately.

4.5 Multi Maps Parameterised by All Trees

After extending our definition of singular maps to trees with many leaves, we are ready to define the collection of singular maps associated to an arbitrary tree. In this section we give the general definition for an arbritrary vertex group, and show that singular maps compose as desired for the classical vertex group. This definition generalises definitions 3.21, 4.1 and 4.13. For clarity we give it in its complete form here.

Recall from section 3.3 that a tree is said to be **augmented** when we have added an additional vertex and edge to the root of the tree, such that the new vertex inherits the former root label, and the old vertex is labelled $G^{\otimes n}$ where n is the number of incoming nodes. We denote the new vertex $\perp$. Also recall that to each internal node, q, of a tree, we can associate the tensor product of the labels of the incoming nodes, and denote it $X_q = X_{q,1} \otimes \cdots \otimes X_{q,n}$. Thus we have $X_\perp = G^{\otimes n}$.

Definition 4.17. *If G is a vertex group and $A_1, \ldots, A_n, B$ are G-modules, then the collection of singular maps associated to an arbitrary tree, p, with n leaves is denoted, $\mathit{Multi}^K_p(A_1, \ldots, A_n; B)$, and is defined as follows:*

- *Let t denote a total ordering, $\perp < root < p_1 < \cdots < p_l$, of the internal vertices of augmented p, compatible with the the partial ordering inherited from the tree structure of p. For each internal vertex, p_i, denote the number of incoming edges n_i. Let $\sigma \in A_{n_i}$ and define an operator taking $G^{\otimes n_i}$-modules to the suitably $G^{\otimes n_i}$-invariant maps:*

$$\mathrm{Sing}_{\sigma,p_i} \cdot = \mathrm{Hom}\Big(X_{p_i,\sigma(1)} \otimes X_{p_i,\sigma(2)}, K \otimes \mathrm{Hom}\Big(X_{p_i,\sigma(3)}, K^{\otimes 2} \otimes \big(\cdots K^{\otimes n_i - 1} \otimes (\cdot) \cdots\big)\Big)\Big). \tag{4.15}$$

  The tensor products of the singularities, K, are taken over appropriate copies of $f_{i,j}$ as in definition 3.3.

- *Let Sing_{p_i} denote the pullback of $\mathrm{Sing}_{\sigma,p_i}$ for all $\sigma \in A_{n_i}$ over $\mathrm{Hom}\Big(X_{p_i}, K^{\otimes \binom{n_i}{2}} \otimes \cdot\Big)$. This can be thought of as giving the multilinear singular maps associated to the flat subtree with root p_i.*

- *Iterating this operator for all internal vertices of p, we have*

$$\mathrm{Ord}_t(A_1, \ldots, A_n, B) = \mathrm{Sing}_{p_l} \cdots \mathrm{Sing}_{p_1} \mathrm{Sing}_{root} \mathrm{Hom}(X_\perp, B).$$

Then $Multi_p^K(A_1,\ldots,A_n;B)$ is defined to be the wide pullback of each Ord_t *for all possible total orderings, t, of of the internal vertices of augmented p, over*

$$\mathrm{Hom}_{G^{\otimes n}}\Big(A_1\otimes\cdots\otimes A_n, Fun_p(G^{\otimes n},B)\Big) \tag{4.16}$$

where $Fun_p(G^{\otimes n},B)$ is the collection of singular functions as defined in [3]. See appendix D for more details.

*We have G-**invariant multilinear singular maps**, $Multi_{G,p}^K$, exactly when* $\mathrm{Hom}(X_\perp,B)$ *is G-invariant.*

Note. *Because limits commute with one another, we could avoid forming the pullback* Sing_{p_i} *by instead defining*

$$\mathrm{Ord}_{t,\sigma_I}(A_1,\ldots,A_n,B)=\mathrm{Sing}_{p_l,\sigma_l}\cdots\mathrm{Sing}_{p_1,\sigma_1}\mathrm{Sing}_{root,\sigma_0}\mathrm{Hom}(X_\perp,B) \tag{4.17}$$

for each $\sigma_i\in A_{n_i}$, and taking the wide pullback for all possible total orderings, t, and permutations $\sigma_i\in A_{n_i}$ for $1\le i\le l$.

Example 4.18. We see automatically that if p is a flat tree, then there is only one total ordering, and so $Multi_p^K(A_1,\ldots,A_n;B)$ reduces to the pullback of $\mathrm{Sing}_{root}\mathrm{Hom}(X_\perp,B)$, over the maps in equation (4.16), which is just definition 4.13.

Theorem 4.19. *If G is a vertex group over a field and $Multi_p^K(A_1,\ldots,A_n;B_1)$, $Multi_q^K(B_1,\ldots,B_m;C)$ are collections of multilinear singular maps, then they compose pointwise to give an element of*

$$Multi_{p\circ q}^K(A_1,\ldots,A_n,B_2\ldots,B_m;C).$$

Note. *Keep in mind that we are composing the trees p and q and not the augmented trees. We only use augmented trees for the purpose of describing their associated singular multi maps.*

Proof. Let $f\in Multi_p^K$ and $g\in Multi_q^K$. We shall show that f and g compose to give an element of $Multi_{p\circ q}^K$. To do so, we select a pullback object $\mathrm{Ord}_{t,\sigma_I}(A_1,\ldots,A_n,B_2\ldots,B_m;C)$ as in equation (4.17). The total ordering of the internal vertices of the tree $p\circ q$ provides a total ordering of both p and q as $root_p<p_1<\cdots<p_k$ and $root_q<q_1<\cdots<q_l$. Such a pullback object also consists of a choice of permutation of labels for each X_{p_i},X_{q_j}, and so we have uniquely determined objects,

$$\begin{aligned}\mathrm{Ord}_{t_p}(A_1,\ldots,A_n,B_1) &= \mathrm{Sing}_{\sigma_k,p_k}\cdots\mathrm{Sing}_{\sigma_1,p_1}\mathrm{Sing}_{\sigma_0,root_p}\mathrm{Hom}(X_{\perp_p},B_1)\\ \mathrm{Ord}_{t_q}(B_1,\ldots,B_m,C) &= \mathrm{Sing}_{\delta_l,q_l}\cdots\mathrm{Sing}_{\delta_1,q_1}\mathrm{Sing}_{\delta_0,root_q}\mathrm{Hom}(X_{\perp_q},C).\end{aligned}$$

Because f and g are contained in the pullbacks, we can regard them as elements of Ord_{t_p} and Ord_{t_q} respectively.

We know that the internal vertices of the composed tree, $p\circ q$ consist exactly of the internal vertices of p, the internal vertices of q, and the root of p. We shall therefore prove that f and g compose to give an element of $\mathrm{Ord}_{t,\sigma_I}$ by induction on the number of internal vertices of $p\circ q$, $k+l+1$. If the composed tree $p\circ q$ has zero internal vertices, then we are composing with a tree of the form $\bullet$ or $\circ$, and it is clear from the discussions of sections 4.1 and 4.2 that maps compose as desired.

Now, without loss of generality we may assume that in the total ordering t, $p_k>q_l$. We need only to prove that there exists a natural map,

$$\begin{aligned}&\Big(\mathrm{Sing}_{\sigma_k,p_k}\cdots\mathrm{Sing}_{\sigma_1,p_1}\mathrm{Sing}_{\sigma_0,root_p}\mathrm{Hom}(X_{\perp_p},B_1)\Big)\otimes\Big(\mathrm{Sing}_{\delta_l,q_l}\cdots\mathrm{Sing}_{\delta_1,q_1}\mathrm{Sing}_{\delta_0,root_q}\mathrm{Hom}(X_{\perp_q},C)\Big)\\ &\longrightarrow \mathrm{Sing}_{\sigma_k,p_k}\bigg(\Big(\mathrm{Sing}_{\sigma_{k-1},p_{k-1}}\cdots\mathrm{Sing}_{\sigma_1,p_1}\mathrm{Sing}_{\sigma_0,root_p}\mathrm{Hom}(X_{\perp_p},B_1)\Big)\otimes\\ &\qquad\Big(\mathrm{Sing}_{\delta_l,q_l}\cdots\mathrm{Sing}_{\delta_1,q_1}\mathrm{Sing}_{\delta_0,root_q}\mathrm{Hom}(X_{\perp_q},C)\Big)\bigg),\end{aligned} \tag{4.18}$$

because on the right hand side, the operator $\text{Sing}_{\sigma_k,p_k}$ is acting on a collection of singular maps associated to a pair of trees with $k+l$ internal vertices, which we know compose by induction. We know that $\Big(\text{Sing}_{\sigma_k,p_k}\cdots\text{Sing}_{\sigma_1,p_1}\text{Sing}_{\sigma_0,root_p}\text{Hom}(X_{\perp_p},B_1)\Big)$ can be evaluated at X_{p_k} (see appendix B), so the left hand side of equation (4.18) can be evaluated at X_{p_k}, giving the inner collection of singular maps on the right hand side. The transpose of this evaluation map provides us with the desired map.

Now that we have seen that the maps f and g compose naturally into each pullback object, Ord_t, our proof will be complete if we show that given two pullback objects, $\text{Ord}_{t_1,\sigma_I}$ and $\text{Ord}_{t_2,\sigma_J}$, the composite of f and g maps through each of them to the same element of

$$\text{Hom}\Big(A_1\otimes\cdots\otimes A_n\otimes B_2\otimes\cdots\otimes B_m, Fun_{p\circ q}(G^{\otimes n+m-1},C)\Big),$$

the collection over which we are pulling back. Denote this collection Z. Since both f and g are elements of a pullback over objects

$$\begin{aligned} X &= \text{Hom}\Big(A_1\otimes\cdots\otimes A_n, Fun_p^G(G^{\otimes n},B_1)\Big)\\ Y &= \text{Hom}\Big(B_1\otimes\cdots\otimes B_m, Fun_q(G^{\otimes m},C)\Big)\end{aligned}$$

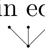
respectively, then if there exists an injective maps from Z to the composite of X and Y, then we know that the composite of f and g maps to the same element of Z, and hence is an element of the pullback. Clearly X and Y compose to give an element of the collection,

$$W=\text{Hom}\Big(A_1\otimes\cdots\otimes A_n\otimes B_2\otimes\cdots\otimes B_m, Fun_p\big(G^{\otimes n}, Fun_q(G^{\otimes m},C)\big)\Big),$$

and because we are working with a vertex group over a field, there is a natural injection from Z to W, so our proof is complete. $\square$

Remark. *Composition can be checked to be associative.*

4.6 Refinement and Maps Between Singular Functions

Now that we have a definition of multilinear singular maps for every possible tree, we consider how these collections are related. Recall that in equation (4.4) we saw that we could map from the collection of singular maps associated to the tree, , to the collections associated to the trees  and  by expanding the singularity, $(x-y)^{-1}$, as a power series in either x or y. We shall now generalise this expansion for an arbitrary vertex group and see that this is an example of a more general phenomenon where our collections of multilinear singular maps are related to one another through canonical maps.

Definition 4.20. *Given any ring of singular functions, K (see definition 2.3), a **refinement for a singularity** is the map,*

$$K \longrightarrow \text{Hom}_G(G,K)$$

which takes any $k\in K$ to the map $f\in\text{Hom}_G(G,K)$ defined by $f(g)=g{\cdot}k$ for any $g\in G$. For any other G-module, A, and G-invariant map $\alpha: A\to G$, we define a refinement for K by composition:

$$K \longrightarrow \text{Hom}_G(G,K) \xrightarrow{\alpha} \text{Hom}_G(A,K).$$

We know that this map is well defined since K is a G-module. It is also obviously a G-linear map. The following examples will show that this is just a generalisation of the idea of power series expansions.

Example 4.21. We saw in section 2.2 that $K=\mathbb{C}[\![x]\!][x^{-1}]$. So a refinement for this singularity is a map:

$$\begin{aligned} \mathbb{C}[\![x]\!][x^{-1}] &\longrightarrow \mathbb{C}[\![x]\!][x^{-1}][\![y]\!]\\ x^k &\mapsto \sum_{i\geq 0}\binom{k}{i}x^{k-i}y^i.\end{aligned}$$

In other words, refinement for this singularity is a map from x^k to $(x+y)^k$ expanded as a power series in the variable y.

Example 4.22. If we compose the refinement map with the antipode map, $S : G \to G$, we have:

$$\begin{array}{ccc} \mathbb{C}\llbracket x\rrbracket[x^{-1}] & \longrightarrow & \mathbb{C}\llbracket x\rrbracket[x^{-1}]\llbracket y\rrbracket \\ x^k & \mapsto & \displaystyle\sum_{i\geq 0}\binom{k}{i}(-1)^i x^{k-i}y^i, \end{array}$$

which is just $(x-y)^k$ expanded as a power series in the variable y.

Example 4.23. If we compose the refinement map with the multiplication map, $\mu : G^{\otimes 2} \to G$, we have:

$$\begin{array}{ccc} \mathbb{C}\llbracket x\rrbracket[x^{-1}] & \longrightarrow & \mathbb{C}\llbracket x\rrbracket[x^{-1}]\llbracket y,z\rrbracket \\ x^k & \mapsto & \displaystyle\sum_{p,q\geq 0}\binom{k}{p+q}\binom{p+q}{p}x^{k-p-q}y^p z^q, \end{array}$$

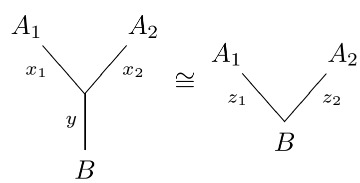
which is just $(x+y+z)^k$ expanded as a power series in the variables y and z.

In order to apply this notion of refinement to our singular multilinear maps, we shall associate a refinement for a singularity to every map between trees in the category of trees.

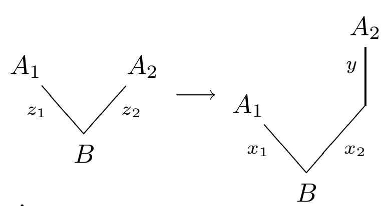
Theorem 4.24. *For G-module, $A_1, \dots, A_n, B$, and any tree p with n leaves, there exists a canonical map $Multi_p^K(A_1,\dots,A_n;B) \to Multi_q^K(A_1,\dots,A_n;B)$ for all trees q which refine to p. This map is given by appropriate refinements of singularities.*

Before we prove this theorem, we give some examples for the classical vertex group.

Example 4.25. We saw in example 4.5 that for the classical vertex group, there is an isomorphism



Example 4.26. Again for the classical vertex group, consider the refinement map



At the level of power series, this is a map,

$$\mathrm{Hom}\Big(A_1{\otimes}A_2, B\llbracket z_1,z_2\rrbracket[(z_1-z_2)^{-1}]\Big) \longrightarrow \mathrm{Hom}\Big(A_2, \mathrm{Hom}\big(A_1, B\llbracket x_1,x_2\rrbracket[(x_1-x_2)^{-1}]\big)\llbracket y\rrbracket\Big)$$

where z_1 is mapped to x_1, z_2 is mapped to x_2+y, and $(z_1-z_2)^{-1}$ is mapped to $(x_1-x_2-y)^{-1}$ and expanded as power series in the variable y. Also we see that automatically $\partial_y = \partial_{x_2}$, which is the same as the G-invariance requirement at the internal node.

Abstractly we have a refinement map

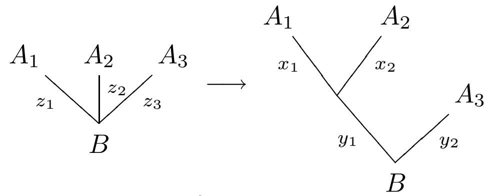
$$\begin{array}{rcl} \mathrm{Hom}\Big(A_1{\otimes}A_2, \mathrm{Hom}(G^{\otimes 2},B){\otimes}K\Big) & \longrightarrow & \mathrm{Hom}\Big(A_2, \mathrm{Hom}\big(A_1{\otimes}G, K{\otimes}\mathrm{Hom}(G^{\otimes 2},B)\big)\Big) \\ & & \cong \mathrm{Hom}\Big(A_1{\otimes}A_2, \mathrm{Hom}\big(G, K{\otimes}\mathrm{Hom}(G^{\otimes 2},B)\big)\Big), \end{array}$$

and so this map reduces to a map

$$\begin{array}{ccc} \mathrm{Hom}(G^{\otimes 2},B){\otimes}K & \longrightarrow & \mathrm{Hom}\big(G, K{\otimes}\mathrm{Hom}(G^{\otimes 2},B)\big) \\ f{\otimes}k & \mapsto & F \end{array}$$

where $F(g) = \sum_{(g)} f(\cdot{\otimes}g_{(1)}\cdot){\otimes}(S(g_{(2)})k)$.

Example 4.27. If we again let G be the classical vertex group and consider the refinement map,



At the level of power series, we have a map of variables,

$$B[\![z_1, z_2, z_3]\!][(z_1 - z_2)^{-1}, (z_1 - z_3)^{-1}, (z_2 - z_3)^{-1}] \longrightarrow B[\![y_1, y_2]\!][(y_1 - y_2)^{-1}][\![x_1, x_2]\!][(x_1 - x_2)^{-1}]$$

where the following are expanded as power series in the variables x_1, x_2.

$$\begin{array}{ll} z_1 \mapsto x_1 + y_1 & (z_1 - z_2)^{-1} \mapsto (x_1 - x_2)^{-1} \\ z_2 \mapsto x_2 + y_1 & (z_1 - z_3)^{-1} \mapsto (x_1 + y_1 - y_2)^{-1} \\ z_3 \mapsto y_2 & (z_2 - z_3)^{-1} \mapsto (x_2 + y_1 - y_2)^{-1}. \end{array}$$

Abstractly, we need to define a map from $Multi^K_{\cdot}$ into

$$\mathrm{Hom}\Big(A_1 \otimes A_2, K \otimes \mathrm{Hom}\Big(G^{\otimes 2} \otimes A_3, K \otimes \mathrm{Hom}(G^{\otimes 2}, B)\Big)\Big).$$

Of the possible pullback objects corresponding to the flat 3 leafed tree, we shall show that there exists a canonical map from

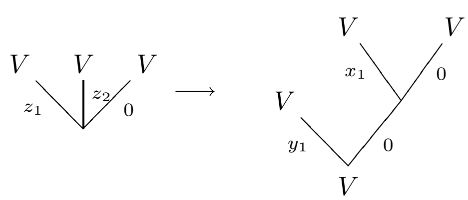
$$\mathrm{Hom}\Big(A_1 \otimes A_2, K \otimes \mathrm{Hom}\Big(A_3, K^{\otimes 2} \otimes \mathrm{Hom}(G^{\otimes 3}, B)\Big)\Big).$$

Since the outer terms are the same we need only define a map

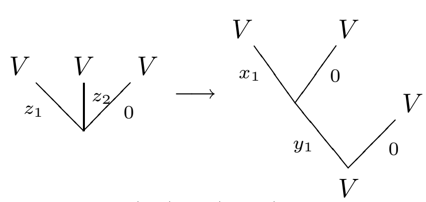
$$\begin{array}{rcl} K^{\otimes 2} \otimes \mathrm{Hom}(G^{\otimes 3}, B) & \longrightarrow & \mathrm{Hom}\Big(G^{\otimes 2}, K \otimes \mathrm{Hom}(G^{\otimes 2}, B)\Big) \\ k_1 \otimes k_2 \otimes f & \mapsto & F, \end{array}$$

where $F(g \otimes h) = \sum_{(g)} \sum_{(h)} \Big[\Big(S(g_{(1)})k_1\Big)\Big(S(h_{(1)})k_2\Big)\Big] \otimes \Big[f\Big(g_{(2)} \cdot \otimes h_{(2)} \cdot \otimes \cdot\Big) \circ (\Delta \otimes 1)\Big]$.

Example 4.28. From the discussion of locality in section 4.3, we see that by setting z_3, x_2 and y_2 equal to zero, the refinement map between G-invariant singular functions,



maps an element $f \in Multi^K_{\cdot}(V, V, V; V)$ to the product of vertex operators, $Y(cdot, y_1)Y(\cdot, x_1)\cdot$. Then the locality condition reduces to the requirement that f be symmetric (this is equivalent to the rationality and commutativity of products of Frenkel, Huang, and Lepowsky, [7]). In fact, the refinement,

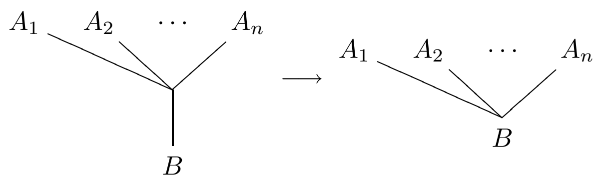


is interpreted as giving a map from f to $Y(Y(\cdot, x_1)\cdot, y_1)\cdot$, and thus formalises the notion of associativity for a vertex algebra as in [7] or [11, section 4.6].

Proof of theorem 4.24. The idea of the proof is that we get a map between pullbacks by showing that there is a map between the spaces over which we are pulling back. There is also a map between the pullback objects, and these maps form a commuting diagram, inducing a map between pullbacks. For more of the abstract details see [15].

Explicitly, a tree p can arise as a refinement of a tree q by a sequence of either replacing nodes of q with edges, or shrinking internal edges of q down to nodes (see appendix A). These two moves can be interchanged, and can be carried out one move at a time. Thus we shall consider them separately. And because we have defined $Multi_p^K$ and $Multi_q^K$ as an iteration of operators associated to each internal node, we need only consider refinements involving flat trees.

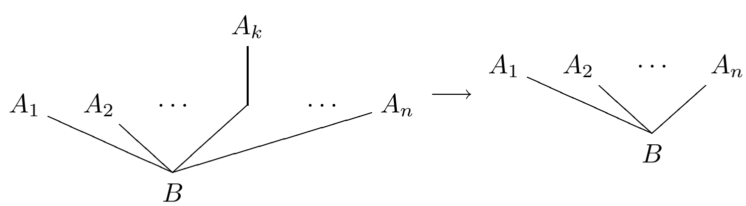
Case 1: We consider the case where p arises as a refinement of q by replacing a node of q with an edge. Taking q to be the flat n leafed tree, we may either replace the root or a leaf to give p. If we replace the root, then we are defining a canonical map



Labelling the interior node of the tree on the left hand side, p_1, we see that an arbitrary pullback object in the definition of $Multi_p^K$, is

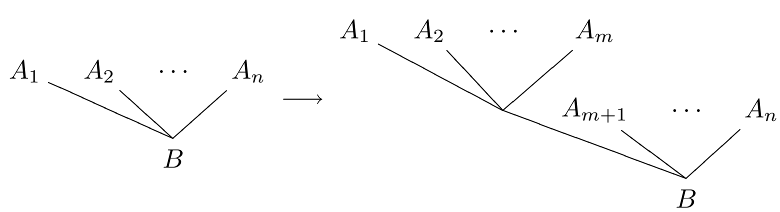
$$\mathrm{Sing}_{\sigma,p_1}\mathrm{Hom}_G\big(G^{\otimes n}, \mathrm{Hom}(G,B)\big),$$

which is manifestly isomorphic to the pullback object, $\mathrm{Sing}_{\sigma,root}\mathrm{Hom}(G^{\otimes n}, B)$, of the right hand side. The same holds for the space over which we pullback, and so we see that in fact this map is an isomorphism. If we instead replace a leaf, we are defining a canonical map



A similar proof shows that this is also an isomorphism.

Case 2: We next consider the case where p arises by shrinking an internal edge of q down to a node. In this case we are defining a canonical map



where, without loss of generality, we have chosen to consider refinement of the first m leaves ($m \leq n$). Labelling the internal node of the tree on the right hand side, q_1, we see that an arbitrary pullback object is of the form

$$\mathrm{Sing}_{\sigma,q_1}\mathrm{Sing}_{\delta,root_q}\mathrm{Hom}(G^{\otimes n-m+1}, B),$$

where $\sigma \in A_m$ and $\delta \in A_{n-m+1}$. We need to show that there exists a natural map from $Multi_p^K$ to this object, so considering an arbitrary element of $Multi_p^K$, we consider it as an element of the pullback object $\mathrm{Sing}_{(\sigma,\delta),root_p}\mathrm{Hom}(G^{\otimes n}, B)$, where we define a permutation, $(\sigma,\delta) \in A_n$:

$$\begin{array}{rcl} (1,\dots,m) & \mapsto & (\sigma(1),\dots,\sigma(m)) \\ (m+1,\dots,n) & \mapsto & (\delta(x),\delta(m+1),\dots,\widehat{\delta(i)},\dots,\delta(n)) \end{array}$$

where x is the placeholder for the internal edge, and where we exclude $\widehat{\delta(i)}$ when i is mapped to x. Notice that we have shifted the domain of δ so that it acts on $(m, \ldots, n)$, and that the permutation, (σ, δ), will arise for $n-m+1$ different choices of δ.

Consider the canonical map we are trying to exhibit:

$$\mathrm{Sing}_{(\sigma,\delta),root_p}\mathrm{Hom}(G^{\otimes n}, B) \longrightarrow \mathrm{Sing}_{\sigma,q_1}\mathrm{Sing}_{\delta,root_q}\mathrm{Hom}(G^{\otimes n-m+1}, B).$$

It will be an identity on the outer terms corresponding to $\mathrm{Sing}_{\sigma,q_1}$. So our problem reduces to showing that there exists a map

$$\begin{aligned} &\mathrm{Hom}\Big(A_{(\sigma,\delta)(m+1)}, K^{\otimes m}\otimes\mathrm{Hom}\Big(A_{(\sigma,\delta)(m+2)}, K^{\otimes m+1}\otimes\cdots\mathrm{Hom}(G^{\otimes n}, B)\Big)\Big) \\ &\longrightarrow \mathrm{Hom}\Bigg(A_{\delta(x)}\otimes A_{\delta(m+1)}, K\otimes\mathrm{Hom}\Big(A_{\delta(m+1)}, K^{\otimes 2}\otimes\mathrm{Hom}\Big(A_{\delta(m+2)}, K^{\otimes 3}\otimes\cdots\mathrm{Hom}(G^{\otimes n-m+1}, B)\Big)\Big)\Bigg), \end{aligned} \tag{4.19}$$

where $A_x = G^{\otimes m}$. We immediately notice that there are fewer copies of the singularity K on the right hand side. This is because when we expand the singularities, we will have maps of the form:

$$K\otimes K \longrightarrow \mathrm{Hom}_G(G, K)\otimes\mathrm{Hom}_G(G, K) \longrightarrow \mathrm{Hom}_G(G^{\otimes 2}, K),$$

where the second arrow is achieved by multiplication in K (e.g., example 4.27). A complicated but routine calculation shows that a succession of maps of this form give the desired map.

□

5 Relaxed Multi Categories

So far in this paper, we have introduced the notion of a vertex group. We then looked at representations of vertex groups, paying special attention to representation of the classical vertex group. We then introduced the notion of multilinear singular maps, and saw that in order for them to compose properly, we needed to parameterise them by trees. Finally we described the canonical maps between singular maps which arise when we refine one tree to another.

In this section we shall be concerned with general properties satisfied by our multilinear singular maps. The work which we have already done in defining, composing and refining the singular maps will be enough to make it clear that we have been working with examples of the following structure.

Definition 5.1. *A* **relaxed multilinear category** *is an ordinary category of R-modules for some ring R with the following additional structure:*

1. **Multi Maps:** *For any $n+1$ objects $A_1, \ldots, A_n, B$ and any n leafed tree, p, there is a collection of multi maps from $A_1, \ldots, A_n$ to B denoted $Multi_p(A_1, \ldots, A_n, B)$. In particular, if $\bullet$ is the unique tree with no internal vertices, then for all objects A, $Multi_\bullet(A, A)$ contains the identity morphism.*

2. **Composition:** *For any $(n+1)$-tuple of multi maps, $Multi_{p_i}(A_{i1}, \ldots, A_{im_i}, B_i)$, $Multi_q(B_1, \ldots, B_n, C)$, $1 \leq i \leq n$ there is a map that defines composition:*

$$\begin{aligned} &Multi_{p_1}(A_{11}, \ldots, A_{1m_1}, B_1)\otimes\ldots\otimes Multi_{p_n}(A_{n1}, \ldots, A_{nm_n}, B_n) \\ &\qquad\qquad\otimes Multi_q(B_1, \ldots, B_n, C) \longrightarrow Multi_{q(p_1,\ldots,p_n)}(A_{11}, \ldots, A_{nm_n}, C), \end{aligned}$$

   where $q(p_1, \ldots, p_n)$ is the tree with $\sum m_i$ leaves formed by gluing the root of each tree p_i to the ith external edge. Composition of maps of this type is associative.

3. **Refinement:** *If p and q are n leaved trees and p is a refinement of q, then for every collection of $n+1$ objects $A_1, \ldots, A_n, B$ there is a map*

$$r_{p,q} : Multi_p(A_1, \ldots, A_n, B) \longrightarrow Multi_q(A_1, \ldots, A_n, B).$$

4. **Linearity:** *The collections of multi maps, $Multi_p(A_1, \dots, A_n, B)$ are R-modules and composition is multilinear.*

The category of representations of a vertex group possesses the structure of a relaxed multilinear category using the G-invariant collections of singular functions. Notice that we need to restrict to G-invariant multi maps in order for the composition condition to hold.

With this categorical structure defined, the next natural step is to define an algebra in a relaxed multilinear category. Recall that we defined an (associative) algebra in a symmetric tensor category in definition 2.9. In a relaxed multilinear category we define an (associative) algebra as follows:

Definition 5.2. *An* **(associative) algebra** *in a relaxed multilinear category, $\mathcal{B}$, consists of an object $B \in \mathcal{B}$ and a collection of maps $\{f_p\} = \{f_p \in Multi^K_{G,p}(B, \dots, B, B) | p \text{ is an } n \text{ leafed tree}, n \in \mathbb{N}\}$. These maps must satisfy the following axioms:*

1. **Composition:** *If $q(p_1, \dots, p_n)$ is the tree formed by gluing the root of each tree p_i to the ith external edge of an n leafed tree, q, (p_i possibly empty), then*

$$f_{q(p_1,\dots,p_n)} = f_q \circ (f_{p_1}, \dots, f_{p_n}). \tag{5.1}$$

2. **Unit:** *The map $f_\circ : R \to B$ (where $\circ$ is the empty tree) defines a unit for the algebra in the sense that for any n leafed tree, p, and any $1 \le k \le n$,*

$$f_p \circ_k f_\circ = f_{p'}$$

where $\circ_k$ denotes composition at the kth leaf of p, and p' is the $n-1$ leaved tree arrived at by removing the kth leaf from p.

3. **Refinement:** *If $p, q \in \mathrm{T}(n)$ and p is a refinement of q, then $r_{p,q}(f_p) = f_q$ where $r_{p,q}$ is the refinement map given by the refinement axiom for a relaxed multilinear category.*

This is an algebra in the sense that each map f_p defines an "n-fold multiplication" for elements of B. For all $n \in \mathbb{N}$ we denote the multilinear map associated to the flat tree with n leaves by f_n. Since composition of multi maps in $\mathcal{B}$ is associative, the associativity of $(B, \{f_p\})$ is a consequence of the composition axiom. Considering $\bullet$, the 1 leafed tree with zero edges, then since $f_p \circ_k f_\bullet = f_p$ and $f_\bullet \circ f_p = f_p$, we see that $f_\bullet = 1_B$. The algebra defined by $(B, \{f_p\})$ is said to be **commutative** if there exists an action of the symmetric group on each of the multilinear maps in $\{f_p\}$.

Example 5.3. Since $f_\bullet$ refines to $f_{|}$, we know that for any $b \in B$, and $g \in G$,

$$f_{|}(b)(g) = gb.$$

Note. *This definition of an algebra is just a functor from the opposite of the category or trees (see appendix A) to $\mathcal{B}$ where each object $p \in \mathcal{T}$ is mapped to an element of $Multi^K_{G,p}$.*

Theorem 5.4. *Given any commutative algebra, $(B, \{f_p\})$, for the classical vertex group, it gives rise to a vertex algebra as in definition 2.1 where B is the space of fields and $|0\rangle = f_0(1)$. The infinitesimal translation operator is $T = D^{(1)}$, and the state-field correspondence is given by $Y(\cdot, x)\cdot = f_2(\cdot, \cdot)|_{y=0} : B \to B[\![x]\!][x^{-1}]$.*

Proof. In order to show that $(B, \{f_p\})$ gives a vertex algebra, we check the axioms for the vertex algebra. The vacuum axioms follow naturally from the discussion of section 4.2, together with previous example, 5.3, giving

$$f_{\vee}(|0\rangle \otimes b) = \sum_{i \ge 0} \frac{T^i}{i!} b y^i = e^{Ty} b.$$

Translation covariance is automatically satisfied because our functions, f_p, are G-invariant. And, we saw that the locality condition is satisfied in section 4.3. $\square$

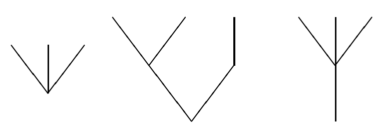
Theorem 5.5. *A vertex algebra (as in definition 2.1) with state space V, defines an algebra in the relaxed multilinear category of representations of the classical vertex group.*

Proof. As in the proof for holomorphic vertex algebras (claim 2.7), the vacuum defines a map $f_\circ$ for the empty tree. We described in example 3.13 how to construct $f_{\vee}$, and section 4.3 showed how how to construct the singular function associated to the flat tree with three leaves. Carrying out a similar process leads to the construction of singular functions associated to all flat trees, and the singular functions associated to trees with internal nodes arise from composition of singular functions associated to flat trees. Closure under refinement follows by construction. $\square$

A Trees

Formally, a **tree** is defined to be a connected oriented planar graph with a finite number of vertices and no circuits. The empty tree is denoted by a circle, $\circ$. All other trees have at least one vertex, and given any such graph we choose a particular vertex which we call the **root** of the tree. For convenience we will always draw the root of the tree at the bottom of the graph. The vertices which are joined to exactly one edge (excluding the root) are called the external vertices or the **leaves** of the tree. All other vertices are called internal vertices. For $n \geq 0$, we let $\mathrm{T}(n)$ denote the collection of all n leafed trees.

We say that a tree has **height** d, if d is the greatest number of edges between the root vertex and a leaf. In (A.1), the first tree has height 1 while the second and third trees have height 2. The unique tree with zero internal vertices is a refinement for every tree in $\mathrm{T}(n)$ and is the only tree with height 1 in that collection. We call trees of height 1 **flat** tress. Each collection of trees $\mathrm{T}(n)$ has a subcollection in which the leaves of each tree are separated from the root vertex by the same number of edges. We denote these **the trees of constant height** $\widetilde{\mathrm{T}}(n)$.



(A.1)

For any two trees p, q with the same number of leaves, we say that p is a **refinement** of q if p arises after a succession (possibly zero) of the following moves:

- an internal edge is shrunk down to a vertex,
- a vertex is replaced by an edge.

By internal edges, we mean those edges that do not end in a leaf. Two trees are considered to be the same exactly when they have the same oriented graph. In (A.1) the first tree is a refinement of the second, and the first and third trees are refinements of one another.

There are a number of (possibly inequivalent) ways of giving the collection of all trees the structure of a category. For our purposes we take the definition due to Tom Leinster [13] which seems to arise most naturally when dealing with higher dimensional categories. (For possibly different definitions see [16] or [5].) In this categorical structure, we define a single morphism from a tree q to a tree p exactly when p is a refinement of q. So in equation (A.1) there exists an arrow from the second tree to the first, and there exists an arrow in each direction between the first tree and the third. Refinement is transitive so morphisms compose and each tree is a refinement of itself so we have identity morphisms. For each $n \geq 0$, $\mathrm{T}(n)$ is a category and so $\mathcal{T}$ is a category with countably many disconnected components. Under this definition, the subcollection of trees of constant height forms a full subcategory of the category of trees, and each $\widetilde{\mathrm{T}}(n)$ is a full subcategory of $\mathrm{T}(n)$.

The category of trees naturally forms an operad, which is the usual multi category of composable trees.

B Evaluation Maps

This appendix is concerned with making clear what types of evaluation maps arise naturally when dealing with collections of multilinear maps of G-modules for an arbitrary vertex group G.

Example B.1. If A and B are G-modules, then we know that there exists a natural evaluation map

$$\mathrm{Hom}(A, B) \otimes B \longrightarrow B.$$

This of course holds for ordinary R-linear maps, $\mathrm{Hom}_R(A, B)$, as well as for G-invariant maps $\mathrm{Hom}_G(A, B)$. In fact, given G modules, $A_1, \ldots, A_n, B$ we have partial evaluation maps

$$\mathrm{Hom}(A_1 \otimes \cdots \otimes A_n, B) \otimes A_1 \otimes A_2 \longrightarrow \mathrm{Hom}(A_3 \otimes \cdots \otimes A_n, B). \tag{B.1}$$

The problem with evaluation arises when we consider singular maps. Recall that for a G-module, B, the collection of singular maps $\mathit{Fun}(G^{\otimes 2}, B)$ was defined to be $\mathrm{Hom}(G^{\otimes 2}, B) \otimes_{f_{1,2}} K$, where the tensor product is taken over H^*. We are tempted to evaluate this map for some $G^{\otimes 2}$,

$$\mathrm{Hom}(G^{\otimes 2}, B) \otimes_{f_{1,2}} K \otimes G^{\otimes 2} \longrightarrow B \otimes K,$$

but no such natural map exists because B does not have the structure of an H^*-module. (Recall that the action of H^* on $\mathrm{Hom}(G^{\otimes 2}, B)$ is on its domain.) The following example makes the problem more explicit:

Example B.2. When G is the classical vertex group, $\mathit{Fun}(G^{\otimes 2}, B) \cong B[\![x, y]\!][(x-y)^{-1}]$, and so an arbitrary element of this collection of maps is,

$$(x-y)^{-k} \sum_{i,j \geq 0} b_{i,j} x^i y^j,$$

for some $b_{i,j} \in B$ and some $k \geq 0$. Naively evaluating this at $D^{(p)} \otimes D^{(q)} \in G^{\otimes 2}$ we have $b_{p,q}(x-y)^{-k} \in B \otimes K$. But if we now rewrite our power series as

$$(x-y)^{-k-1} \sum_{i,j \geq 0} b_{i,j}(x^{i+1} y^j + x^i y^{j+1},$$

we see that this evaluates to $(x-y)^{-k-1}(b_{p-1,q} + b_{p,q-1})$ for $p, q \geq 1$. Thus we see that the same power series seems to have evaluated to two different solutions. We still might think that the situation could be reconciled by setting the two evaluation equal to one another, giving

$$b_{p-1,q} + b_{p,q-1} = b_{p,q}(x-y).$$

But this is just a restriction on the form of our power series, it does not define an action of H^* on B as we might have hoped.

Now that we see that there are evaluations which we can and can not make, we would like to know how to evaluate multilinear singular maps, $\mathit{Multi}^{K}_{\curlyvee}(A_1, \ldots, A_n; B)$, for an arbitrary tree, p. Since these are defined as pullbacks over $\mathrm{Hom}_{G^{\otimes n}}\Big(A_1 \otimes \cdots \otimes A_n, \mathit{Fun}_p(G^{\otimes n}, B)\Big)$, we know that they can be evaluated partially or completely for all $A_1, \ldots, A_n$ as in equation (B.1). In fact, the singular maps, $\mathrm{Sing}_\sigma(A_1, \ldots, A_n; \mathrm{Hom}(G^{\otimes n}, B))$ of equation (4.9) can also be evaluated, because although there are many copies of the singularity K among the A_i, the tensoring over H^* is with the inner copy of $G^{\otimes n}$ (this can be realised as a suitable quotient), and so we could for example evaluate the following at $A_{\sigma(2)}, A_{\sigma(3)}$:

$$\begin{aligned}\mathrm{Hom}\Big(A_{\sigma(1)} \otimes A_{\sigma(2)}, K \otimes \mathrm{Hom}\Big(A_{\sigma(3)}, K^{\otimes 2} \otimes \big(\cdots K^{\otimes n-1} \otimes \mathrm{Hom}(G^{\otimes n}, B) \cdots\big)\Big)\Big) \\ \longrightarrow \mathrm{Hom}\Big(A_{\sigma(1)}, K \otimes \mathrm{Hom}\Big(A_{\sigma(4)}, K^{\otimes 2} \otimes K^{\otimes 3} \otimes \big(\cdots K^{\otimes n-1} \otimes \mathrm{Hom}(G^{\otimes n}, B) \cdots\big)\Big)\Big).\end{aligned} \tag{B.2}$$

C Selected Results from Axiomatic Vertex Algebras

In this appendix we gather a few facts about ordinary axiomatic vertex algebras which we have used throughout this paper. For more details see [11].

Let $(V, Y(\cdot, x)\cdot, T, |0\rangle)$ be a vertex algebra. This first lemma is a consequence of the vacuum axioms:

Lemma C.1. *For any $a \in V$, $Y(a,x)|0\rangle = \sum_{i\geq 0} \frac{T^i}{i!} a x^i = e^{xT} a$.*

Proof. Since $Y(a,x)|0\rangle$ can be evaluated at zero, we know that it has no singularities. By the first vacuum axiom we know that $T|0\rangle = 0$, so the translation covariance axiom says that for any $n \geq 0$, $(\partial_x)^n Y(a,x)|0\rangle = \frac{T^n}{n!} Y(a,x)|0\rangle$. Letting $Y(a,x)|0\rangle = \sum_{i\geq 0} a_i x^i$, and evaluating this at zero we have $a_n = \frac{T^n}{n!} a$ and the lemma is proved. □

The next lemma uses this result, and is a consequence of the locality axiom.

Lemma C.2 (Quasisymmetry). *For any $a, b \in V$ we have $Y(a,x)b = e^{xT} Y(b, -x)a$.*

Proof. Recall that the translation covariance axiom says that $Y(a,x)T = (T - \partial_x)Y(a,x)$. So we have

$$\begin{aligned} Y(b,y)e^{xT}a &= \sum_{i\geq 0} Y(b,y)\frac{T^i}{i!}a \\ &= \sum_{p,q\geq 0} (-1)^q \frac{T^p}{p!}\frac{\partial_y^q}{q!} x^{p+q} Y(b,y)a \\ &= \sum_{p\geq 0} \frac{T^p}{p!} x^p Y(b, y-x)a \\ &= e^{xT} Y(b, y-x)a. \end{aligned}$$

From the locality axiom we know that for some $N \gg 0$ the following holds

$$(x-y)^N Y(a,x)Y(b,y)|0\rangle = (x-y)^N Y(b,y)Y(a,x)|0\rangle.$$

Combining this with the previous lemma, it just says that

$$\begin{aligned} (x-y)^N Y(a,x)e^{yT}b &= (x-y)^N Y(b,y)e^{xT}a \\ &= (x-y)^N e^{xT} Y(b, y-x)a. \end{aligned}$$

Setting y equal to zero we have $x^N Y(a,x)b = x^N e^{xT} Y(b,-x)a$, and since this is an equality of Laurent series, we may divide by x^N to give our result. □

D Borcherds' Singular maps parameterised by trees

This appendix provides a review of the singular multilinear maps defined in [3]. In that paper, the definitions which follow were intended only for trees of constant height (i.e., trees whose root is separated from each leaf by the same number of edges), which are equivalent to the **sieves** defined in that paper. Here we extend the definition of that paper to all trees. Our definition for multilinear singular maps uses these singular maps as a base over which we pullback to get a more specific collection of multilinear singular maps.

Definition D.1. *If p is a tree with n leaves and height d, then we define the* ***space of singular functions of type p from $G^{\otimes n}$ to B****, written as $Fun_p(G^{\otimes n}, B)$, recursively on height. If the height of p is one, then p is the flat tree with zero internal vertices, and we define $Fun_p(G^{\otimes n}, B)$ to be $Fun(G^{\otimes n}, B)$ as in definition 3.3. If $d > 1$, then let q be the tree of height $d-1$ with m leaves, formed by removing the s edges which are separated from the root by $d-1$ edges. We define $Fun_p(G^{\otimes n}, B)$ as follows:*

1. *Begin by forming the collection of $G^{\otimes m}$-invariant maps $\mathrm{Hom}_{G^{\otimes m}}(G^{\otimes s}, Fun_q(G^{\otimes m}, B))$. We make $G^{\otimes s}$ into a $G^{\otimes m}$-module by first noticing that each leaf $0 \leq l \leq m$ of q is joined to k_l edges of p. So the lth entry of $G^{\otimes m}$ acts diagonally on the corresponding k_l entries of $G^{\otimes n}$.*

2. *Localise* $\mathrm{Hom}_{G^{\otimes m}}(G^{\otimes n}, Fun_q(G^{\otimes m}, B))$ *at all pairs* (i,j) *of the* k_l *entries of* $G^{\otimes n}$ *which are joined to each leaf* $1 \leq l \leq m$. *This collection of singular maps is* $Fun_p(G^{\otimes n}, B)$.

Note. *This difference between this definition and the one given in [3] is that in step 1 above, we require the maps from* $G^{\otimes n}$ *to* $Fun_q(G^{\otimes m}, B)$ *to be* G*-invariant at each leaf of* q. *We will see in the following examples that this definition gives the results put forward in that paper. It is our belief that this assumption of* $G^{\otimes m}$*-invariance is presumed in the definition given in [3].*

The definition is made clear by working out a number of examples.

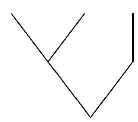
Example D.2. Let G be a vertex group and B be a G-module. We begin by looking at the space of singular functions from G to B, parameterised by non-branching trees. For the tree, p_1 consisting of just one edge, we know that $Fun_{p_1}(G,B) = Fun(G,B) \cong \mathrm{Hom}(G,B)$. Using the given recursion definition, if we let p_i be the tree of height i, then $Fun_{p_2}(G,B) = \mathrm{Hom}_G(G, \mathrm{Hom}(G,B)) \cong \mathrm{Hom}(G,B)$, and so for any $i \geq 1$, $Fun_{p_i}(G,B) \cong \mathrm{Hom}(G,B)$.

Example D.3. Similarly, if we have an n-leafed tree p, and if q is the tree formed from pasting the root of p onto the end of the tree, $|$, then $Fun_q(G^{\otimes n}, B)$ is the module $\mathrm{Hom}_G(G^{\otimes n}, \mathrm{Hom}(G,B))$ localised at all pairs (i,j). But we know $\mathrm{Hom}_G(G^{\otimes n}, \mathrm{Hom}(G,B)) \cong \mathrm{Hom}_R(G^{\otimes n}, B)$, and so $Fun_q(G^{\otimes n}, B) \cong Fun_p(G^{\otimes n}, B)$.

Example D.4. Let t_3 be the unique 3-leafed tree of height 1. We saw in example 3.8 that for the classical vertex group,

$$\begin{aligned} Fun_{t_3}(G^{\otimes 3}, B) &\cong Fun(G^{\otimes 3}, B) \\ &\cong B[\![x_1, x_2, x_3]\!][(x_1 - x_2)^{-1}, (x_1 - x_3)^{-1}(x_2 - x_3)^{-1}]. \end{aligned}$$

If we let the following tree be denoted l:



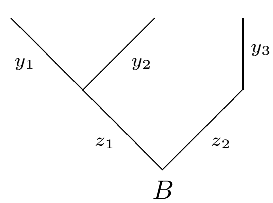
then we have that $Fun_l(G^{\otimes 3}, B)$ is the following module localised at $(1,2)$:

$$\mathrm{Hom}_{G^{\otimes 2}}(G^{\otimes 3}, Fun_{t_2}(G^{\otimes 2}, B)) \cong \mathrm{Hom}_{G^{\otimes 2}}(G^{\otimes 3}, B[\![z_1, z_2]\!][(z_1 - z_2)^{-1}]).$$

Localising, we can identify this module with a subset of the collection of formal power series

$$B[\![z_1, z_2]\!][(z_1 - z_2)^{-1}][\![y_1, y_2, y_3]\!][(y_1 - y_2)^{-1}]. \tag{D.1}$$

The $G^{\otimes 2}$-invariance can be realised as the restriction to power series which satisfy $\partial_{z_1} = \partial_{y_1} + \partial_{y_2}$ and $\partial_{z_2} = \partial_{y_3}$. We represent this module pictorially as the labelled tree,



where at each internal node we have equalised over an action of G on the incoming and outgoing edges. These differential equations give conditions which allow us to rewrite this power series module as a power series of only three variables. Either of the changes of variables given in equations (3.5)-(3.7) work, and so we may made the identification

$$\begin{aligned} Fun_l(G^{\otimes 3}, B) &\cong B[\![X_1, X_3]\!][(X_1 - X_3)^{-1}][\![(X_1 - X_2)]\!][(X_1 - X_2)^{-1}] \\ &\cong B[\![X_1, X_3]\!][(X_1 - X_3)^{-1}][\![X_2]\!][(X_1 - X_2)^{-1}]. \end{aligned}$$

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